EXERCISE SOLUTIONS, LECTURES 15-20

Contents

15. Directional derivatives and the gradient

Exercise 1. Find the gradient of f.

(1) $f(x,y) = 3x^2y - xy^3$
(2) $f(x,y) = \frac{x}{x+y}$
(3) $f(x,y) = \sqrt{x^2+y^2}$
(4) $f(x,y) = x\ln(x) + y\ln(y)$
(5) $f(x,y) = e^{x\sin(y)}$
(6) $f(x,y,z) = \frac{x}{y+z}$
(7) $f(x,y,z) = x\ln(yz)$
(8) $f(x,y,z) = xyze^{xyz}$

Solution. (1)

$$\nabla f(x,y) = \langle 6xy - y^3, 3x^2 - 3xy^2\rangle$$

(2) We write this as $f(x,y) = x(x+y)^{-1}$. Then

$$f_x(x,y) = (x+y)^{-1} + x\frac{\partial}{\partial x}\left((x+y)^{-1}\right) = \frac{1}{x+y} - x(x+y)^{-2} = \frac{1}{x+y} - \frac{x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$f_y(x,y) = -x(x+y)^{-2} = -\frac{x}{(x+y)^2}$$

So

$$\nabla f(x,y) = \langle \frac{y}{(x+y)^2}, -\frac{x}{(x+y)^2}\rangle$$

(3) We write this as $f(x,y) = (x^2+y^2)^{1/2}$. Then

$$f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

So

$$\nabla f(x,y) = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$$

(4) We have

$$f_x(x,y) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$
$$f_y(x,y) = \ln(y) + y \cdot \frac{1}{y} = \ln(y) + 1$$

So

$$\nabla f(x,y) = \langle \ln(x)+1, \ln(y)+1 \rangle$$

(5) We have by Chain Rule

$$f_x(x,y) = e^{x\sin(y)}\sin(y), \quad f_y(x,y) = e^{x\sin(y)}x\cos(y),$$

so

$$\nabla f(x,y) = \langle e^{x\sin(y)}\sin(y), e^{x\sin(y)}x\cos(y) \rangle$$

(6) We write this as $f(x,y,z) = x(y+z)^{-1}$. We have

$$f_x(x,y,z) = (y+z)^{-1} = \frac{1}{y+z}$$
$$f_y(x,y,z) = -x(y+z)^{-2} = -\frac{x}{(y+z)^2}$$
$$f_z(x,y,z) = -x(y+z)^{-2} = -\frac{x}{(y+z)^2}$$

So

$$\nabla f(x,y,z) = \langle \frac{1}{y+z}, -\frac{x}{(y+z)^2}, -\frac{x}{(y+z)^2} \rangle$$

(7) We have

$$f_x(x,y,z) = \ln(yz), \quad f_y(x,y,z) = x\frac{1}{yz} \cdot z = \frac{x}{y}, \quad f_z(x,y,z) = x\frac{yz}{\cdot}y = \frac{x}{z}$$

(8) We have

$$f_x(x,y,z) = yze^{xyz} + xyze^{xyz} \cdot (yz) = (yz + xy^2z^2)e^{xyz}$$
$$f_y(x,y,z) = xze^{xyz} + xyze^{xyz} \cdot (xz) = (xz + x^2yz^2)e^{xyz}$$
$$f_z(x,y,z) = xye^{xyz} + xyze^{xyz} \cdot (xy) = (xy + x^2y^2z)e^{xyz}$$

So

$$\nabla f(x,y,z) = \langle (yz + xy^2z^2)e^{xyz}, (xz + x^2yz^2)e^{xyz}, (xy + x^2y^2z)e^{xyz} \rangle$$

□

Exercise 2. Find the directional derivative.

(1) $D_{\vec{u}}f(1,1)$, where $f(x,y) = x^2 + y^2$ and $\vec{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

(2) $D_{\vec{u}}f(3,0)$, where $f(x,y) = x^2e^y$ and $\vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.

Solution. (1) Note $D_{\vec{u}}f(1,1) = \vec{u} \cdot \nabla f(1,1)$. We have

$$f_x(x,y) = 2x, \quad f_y(x,y) = 2y,$$

so

$$\nabla f(1,1) = \langle 2, 2 \rangle$$

so

$$D_{\vec{u}}f(1,1) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \cdot \langle 2, 2 \rangle = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0$$

(2) Note that $D_{\vec{u}}f(3,0) = \vec{u} \cdot \nabla f(3,0)$. We have

$$f_x(x,y) = 2xe^y, \quad f_y(x,y) = x^2 e^y,$$

so

$$\nabla f(3,0) = \langle 6, 9 \rangle$$

so

$$D_{\vec{u}}f(3,0) = \langle \frac{3}{5}, -\frac{4}{5} \rangle \cdot \langle 6, 9 \rangle = \frac{18}{5} - \frac{36}{5} = -\frac{18}{5}$$

□

Exercise 3. Find the maximum rate of increase of f at the given point, and the direction in which it occurs.

(1) $f(x,y) = \sin(xy)$ at $(1,0)$.
(2) $f(x,y) = 2xy^2 + xy^3$ at $(1,2)$.
(3) $f(x,y,z) = xyz^2 + x^2y^2$ at $(1,0,-1)$

Solution. (1) The direction of maximum rate of increase is the unit vector in the direction of gradient, $\nabla f(1,0)$. Note

$$f_x(x,y) = y\cos(xy), \quad f_y(x,y) = x\cos(xy),$$

so

$$\nabla f(1,0) = \langle 0, 1 \rangle$$

Since this is already a unit vector, the direction of maximum rate of increase is $\langle 0, 1 \rangle$. The maximum rate of increase is $|\nabla f(1,0)| = 1$.

(2) The direction of maximum rate of increase is the unit vector in the direction of gradient, $\nabla f(1,2)$. Note

$$f_x(x,y) = 2y^2 + y^3, \quad f_y(x,y) = 4xy + 3xy^2$$

so

$$\nabla f(1,2) = \langle 2 \cdot 2^2 + 2^3, 4 \cdot 2 + 3 \cdot 2^2 \rangle = \langle 16, 20 \rangle$$

Thus the direction of maximum rate of increase is

$$\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle 16, 20 \rangle}{\sqrt{16^2 + 20^2}} = \frac{\langle 16, 20 \rangle}{\sqrt{656}} = \frac{\langle 4, 5 \rangle}{\sqrt{41}} = \langle \frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}} \rangle$$

The maximum rate of increase is $|\nabla f(1,2)| = 4\sqrt{41}$.

(3) The direction of maximum rate of increase is the unit vector in the direction of gradient, $\nabla f(1,0,-1)$. Note

$$f_x(x,y,z) = yz^2 + 2xy^2, \quad f_y(x,y,z) = xz^2 + 2x^2y, \quad f_z(x,y,z) = 2xyz$$

so

$$\nabla f(1,0,-1) = \langle 0,1,0\rangle$$

This is already a unit vector, so $\langle 0,1,0\rangle$ is the direction of the maximum rate of increase. The maximum rate of increase is $|\nabla f(1,0,-1)| = 1$.

$\square$

Exercise 4. Find the tangent plane.

(1) Tangent plane to $xyz = 6$ at $(1,2,3)$
(2) Tangent plane to $x + y + z = e^{xyz}$ at $(0,0,1)$
(3) Tangent plane to $x^4 + y^4 + z^4 = 3x^2y^2z^2$ at $(1,1,1)$

Solution. (1) The surface is the level surface $f(x,y,z) = 6$ where $f(x,y,z) = xyz$. The equation of the tangent plane is

$$f_x(1,2,3)(x-1) + f_y(1,2,3)(y-2) + f_z(1,2,3)(z-3) = 0$$

Note that

$$f_x(x,y,z) = yz, \quad f_y(x,y,z) = xz, \quad f_z(x,y,z) = xy$$

so

$$f_x(1,2,3) = 6, \quad f_y(1,2,3) = 3, \quad f_z(1,2,3) = 2$$

so the tangent plane has equation

$$6(x-1) + 3(y-2) + 2(z-3) = 0,$$

or

$$6x + 3y + 2z = 18$$

(2) The surface is the level surface $f(x,y,z) = 0$ where $f(x,y,z) = x + y + z - e^{xyz}$. The equation of the tangent plane is

$$f_x(0,0,1)(x-0) + f_y(0,0,1)(y-0) + f_z(0,0,1)(z-1) = 0$$

Note that

$$f_x(x,y,z) = 1 - yze^{xyz}, \quad f_y(x,y,z) = 1 - xze^{xyz}, \quad f_z(x,y,z) = 1 - xye^{xyz}$$

so

$$f_x(0,0,1) = 1, \quad f_y(0,0,1) = 1, \quad f_z(0,0,1) = 1$$

so the tangent plane has equation

$$x + y + z - 1 = 0$$

(3) The surface is the level surface $f(x,y,z) = 0$ where $f(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. The equation of the tangent plane is

$$f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) = 0$$

Note that

$$f_x(x,y,z) = 4x^3 - 6xy^2z^2, \quad f_y(x,y,z) = 4y^3 - 6x^2yz^2, \quad f_z(x,y,z) = 4z^3 - 6x^2y^2z$$

so

$$f_x(1,1,1) = -2, \quad f_y(1,1,1) = -2, \quad f_z(1,1,1) = -2$$

so the tangent plane has equation

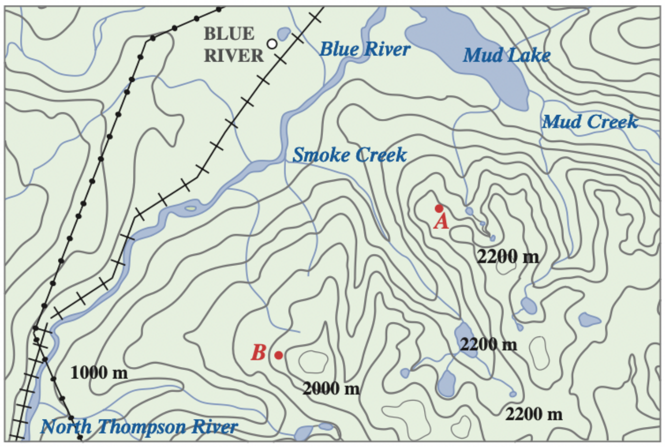
$$-2(x-1) - 2(y-1) - 2(z-1) = 0,$$

or

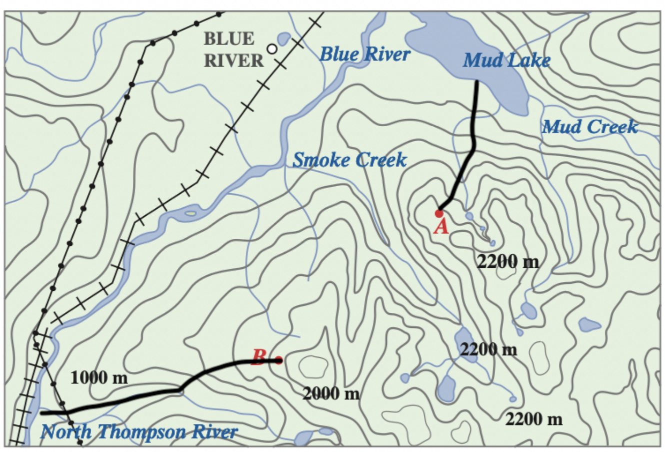
$$-2x - 2y - 2z = -6$$

□

Exercise 5. Shown is a topographic map of Blue River Pine Provincial Park in British Columbia. Draw curves of steepest descent from point A (descending to Mud Lake) and from point B.



Solution. You follow the negative gradient vectors. The picture is just an approximation, so there might be certain inaccurancies in the drawing.



□

16. Local maxima and minima, critical points

Exercise 1. Find the critical points and use the Second Derivative Test to determine whether they are local minima, local maxima or saddle points.

(1) $f(x, y) = xy - 2x - 2y - x^2 - y^2$
(2) $f(x, y) = y(e^x - 1)$
(3) $f(x, y) = 2 - x^4 + 2x^2 - y^2$
(4) $f(x, y) = (6x - x^2)(4y - y^2)$
(5) $f(x, y) = (x^2 + y^2)e^{-x}$
(6) $f(x, y) = \sin x \sin y$, in $-\pi < x < \pi$ and $-\pi < y < \pi$
(7) $f(x, y) = y^2 - 2y \cos x$, in $-1 \leq x \leq 7$ and $-3 \leq y \leq 3$
(8) $f(x, y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2$
(9) $f(x, y) = 3xe^y - x^3 - e^{3y}$

Solution. (1) We first find the critical points. Note

$$f_x(x, y) = y - 2 - 2x, \quad f_y(x, y) = x - 2 - 2y$$

So if (x, y) is a critical point, this means

$$y - 2 - 2x = 0, \quad x - 2 - 2y,$$

or

$$y = 2 + 2x, \quad x = 2 + 2y.$$

Plugging $x = 2 + 2y$ into $y = 2 + 2x$, we get

$$y = 2 + 2(2 + 2y) = 6 + 4y,$$

or $6 = -3y$, or $y = -2$. From this, we get $x = 2 - 4 = -2$. So there is only one critical point, $(-2, -2)$.

To use the Second Derivative Test, we need to compute what $D(x, y)$ is. Note that

$$f_{xx}(x, y) = -2, \quad f_{xy}(x, y) = 1, \quad f_{yy}(x, y) = -2,$$

so

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}(x, y)^2 = 4 - 1 = 3.$$

This is always positive. Note also that f_{xx} is always $-2 < 0$, so any critical point has to be local maximum.

(2) We first find the critical points. Note

$$f_x(x, y) = ye^x, \quad f_y(x, y) = e^x - 1$$

so if (x, y) is a critical point, this means

$$ye^x = 0, \quad e^x - 1 = 0.$$

Since $ye^x = 0$ means $y = 0$ or $e^x = 0$, and since e^x is never zero, this means $y = 0$. The second equation means $e^x = 1$, or $x = \ln(1) = 0$. Thus there is one critical point, $(0, 0)$.

To use the Second Derivative Test, we need to compute what $D(x, y)$ is. Note that

$$f_{xx}(x, y) = ye^x, \quad f_{xy}(x, y) = e^x, \quad f_{yy}(x, y) = 0,$$

so

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}(x, y)^2 = -e^{2x}.$$

This is always negative, so any critical point is a saddle point.

(3) We first find the critical points. Note that

$$f_x(x,y) = -4x^3 + 4x, \quad f_y(x,y) = -2y.$$

So if (x,y) is a critical point, this means

$$-4x^3 + 4x = 0, \quad -2y = 0.$$

So first of all $y = 0$, and we have $-4x(x-1)(x+1) = 0$. Thus x could be either $0, -1$ or 1. The critical points are $(0,0)$, $(-1,0)$ and $(1,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = -12x^2 + 4, \quad f_{xy}(x,y) = 0, \quad f_{yy}(x,y) = -2,$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = 24x^2 - 8$$

Thus

$$D(0,0) = -8 < 0$$

which means that $(0,0)$ is a saddle point. Also,

$$D(-1,0) = 24 - 8 > 0, \quad f_{xx}(-1,0) = -12 + 4 < 0,$$

which means that $(-1,0)$ is a local maximum.

$$D(1,0) = 24 - 8 > 0, \quad f_{xx}(1,0) = -12 + 4 < 0,$$

which means that $(1,0)$ is a local maximum.

(4) We first find the critical points. Note that

$$f_x(x,y) = (6-2x)(4y-y^2), \quad f_y(x,y) = (6x-x^2)(4-2y),$$

so if (x,y) is a critical point, it means

$$(6-2x)(4y-y^2) = 0, \quad (6x-x^2)(4-2y) = 0.$$

The first equation means that either $6 - 2x = 0$ or $4y - y^2 = 0$. Note also that $6 - 2x = 0$ means $x = 3$, and $4y - y^2 = 0$ means either $y = 0$ or $y = 4$. So the first requirement is either $x = 3$, $y = 0$ or $y = 4$.

The second equation means that either $6x - x^2 = 0$ or $4 - 2y = 0$. Note also that $6x - x^2 = 0$ means either $x = 0$ or $x = 6$, and $4 - 2y = 0$ means $y = 2$. So the second requirement is either $x = 0$, $x = 6$ or $y = 2$.

So a pair (x,y) satisfying the two requirements are as follows, following the first requirement first:

- If $x = 3$, then out of the three possible outcomes of the second requirement, $x = 0$, $x = 6$ or $y = 2$, the only possibility is $y = 2$, so $(3,2)$.
- If $y = 0$, then out of the three possible outcomes of the second requirement, $x = 0$, $x = 6$ or $y = 2$, it could possibly be either $x = 0$ or $x = 6$, so $(0,0)$ or $(6,0)$.
- If $y = 4$, then out of the three possible outcomes of the second requirement, $x = 0$, $x = 6$ or $y = 2$, it could possibly be either $x = 0$ or $x = 6$, so $(0,4)$ or $(6,4)$.

So the critical points are $(3,2)$, $(0,0)$, $(6,0)$, $(0,4)$ and $(6,4)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = -2(4y-y^2), \quad f_{xy}(x,y) = (6-2x)(4-2y), \quad f_{yy}(x,y) = -2(6x-x^2)$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y) = 4(4y-y^2)(6x-x^2) - (6-2x)^2(4-2y)^2$$

We apply the Second Derivative Test to the five critical points.

- If $(x,y) = (3,2)$, then
$$D(3,2) = 4(8-4)(18-9) - 0 > 0, \quad f_{xx}(x,y) = -2(8-4) < 0,$$
so $(3,2)$ is a local maximum.
- If $(x,y) = (0,0)$, then
$$D(0,0) = 0 - 6^2 4^2 < 0,$$
so $(0,0)$ is a saddle point.
- If $(x,y) = (6,0)$, then
$$D(6,0) = 0 - (6-12)^2 4^2 < 0,$$
so $(6,0)$ is a saddle point.
- If $(x,y) = (0,4)$, then
$$D(0,4) = 0 - 6^2(4-8)^2 < 0,$$
so $(0,4)$ is a saddle point.
- If $(x,y) = (6,4)$, then
$$D(6,4) = 0 - (6-12)^2(4-8)^2 < 0,$$
so $(6,4)$ is a saddle point.

(5) We first find the critical points. Note that
$$f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x} = (2x - x^2 - y^2)e^{-x}, \quad f_y(x,y) = 2ye^{-x},$$
so if (x,y) is a critical point, it means that
$$(2x - x^2 - y^2)e^{-x} = 0, \quad 2ye^{-x} = 0.$$
Since e^{-x} is never zero, this means
$$2x - x^2 - y^2 = 0, \quad 2y = 0.$$
So $y = 0$, and $2x - x^2 = 0$, which means either $x = 0$ or $x = 2$. So the critcial points are $(0,0)$ and $(2,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that
$$f_{xx}(x,y) = (2-2x)e^{-x} - (2x - x^2 - y^2)e^{-x} = (2 - 4x + x^2 + y^2)e^{-x},$$
$$f_{xy}(x,y) = -2ye^{-x},$$
$$f_{yy}(x,y) = 2e^{-x}.$$
So
$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = (4 - 8x + 2x^2 + 2y^2)e^{-2x} - 4y^2e^{-2x}$$

- For the critical point $(0,0)$, we have
$$D(0,0) = 4 > 0, \quad f_{xx}(0,0) = 2 > 0,$$
so $(0,0)$ is a local minimum.

- For the critical point $(2,0)$, we have
$$D(2,0) = (4-16+8)e^{-4} < 0,$$
so $(2,0)$ is a saddle point.

(6) We first find the critical points. Note that
$$f_x(x,y) = \cos x \sin y, \quad f_y(x,y) = \sin x \cos y,$$
so if (x,y) is a critical point, it means
$$\cos x \sin y = 0, \quad \sin x \cos y = 0.$$
So the first requirement is either $\cos x = 0$ or $\sin y = 0$, and the second requirement is either $\sin x = 0$ or $\cos y = 0$.
- If $\cos x = 0$, then $\sin x \neq 0$, so $\cos y = 0$.
- If $\sin y = 0$, then $\cos y \neq 0$, so $\sin x = 0$.

So (x,y) is a critical point if either $\cos x = \cos y = 0$ or $\sin x = \sin y = 0$.
- If $\cos x = \cos y = 0$, then it means x, y are either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. So the critical points coming out of this possibility are $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{2})$.
- If $\sin x = \sin y = 0$, then it means x, y are both 0, so the critical point coming out of this possibility is $(0,0)$.

So the critcial points in the domain are $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(0,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that
$$f_{xx}(x,y) = -\sin x \sin y, \quad f_{xy}(x,y) = \cos x \cos y, \quad f_{yy}(x,y) = -\sin x \sin y,$$
so
$$D(x,y) = \sin^2 x \sin^2 y - \cos^2 x \cos^2 y.$$
- For the critical point $(-\frac{\pi}{2}, -\frac{\pi}{2})$, we have
$$D(-\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \quad f_{xx}(-\frac{\pi}{2}, -\frac{\pi}{2}) = -1 < 0,$$
so $(-\frac{\pi}{2}, -\frac{\pi}{2})$ is a local maximum.
- For the critical point $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have
$$D(-\frac{\pi}{2}, \frac{\pi}{2}) = 1 > 0, \quad f_{xx}(-\frac{\pi}{2}, \frac{\pi}{2}) = 1 > 0,$$
so $(-\frac{\pi}{2}, \frac{\pi}{2})$ is a local minimum,
- For the critical point $(\frac{\pi}{2}, -\frac{\pi}{2})$, we have
$$D(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \quad f_{xx}(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0,$$
so $(\frac{\pi}{2}, -\frac{\pi}{2})$ is a local minimum.
- For the critical point $(\frac{\pi}{2}, -\frac{\pi}{2})$, we have
$$D(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \quad f_{xx}(\frac{\pi}{2}, -\frac{\pi}{2}) = -1 < 0,$$
so $(\frac{\pi}{2}, -\frac{\pi}{2})$ is a local maximum.
- For the critical point $(0,0)$, we have
$$D(0,0) = -1 < 0,$$
so $(0,0)$ is a saddle point.

(7) We first find the critical points. Note that

$$f_x(x, y) = 2y \sin x, \quad f_y(x, y) = 2y - 2\cos x,$$

so if (x, y) is a critical point, we have

$$2y \sin x = 0, \quad 2y - 2\cos x = 0.$$

So $y = \cos x$, and either $y = 0$ or $\sin x = 0$. If $y = 0$, then $\cos x = 0$, which means that $x = \cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$. Since $-\frac{\pi}{2} \sim -1.57$, $\frac{3\pi}{2} \sim 4.71$, $\frac{5\pi}{2} \sim 7.85$, the points in the range $-1 \leq x \leq 7$ are $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. If $\sin x = 0$, then $\cos x$ could be either 1 or -1, so $y = 1$ or -1. Note also that $\sin x = 0$ in the range $-1 \leq x \leq 7$ means $x = 0, \pi$ or 2π, because $3\pi \sim 9.42 > 7$ and $-\pi \sim -3.14 < -1$. So the critical points are $(\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$, $(0, 1)$, $(0, -1)$, $(\pi, 1)$, $(\pi, -1)$, $(2\pi, 1)$, $(2\pi, -1)$.

To use the Second Derivative Test, we need to compute what $D(x, y)$ is. Note that

$$f_{xx}(x, y) = 2y \cos x, \quad f_{xy}(x, y) = 2 \sin x, \quad f_{yy}(x, y) = 2$$

So

$$D(x, y) = 4y \cos x - 4 \sin^2 x.$$

- For the critical point $(\frac{\pi}{2}, 0)$, we have

  $$D(\frac{\pi}{2}, 0) = -4 < 0,$$

  so $(\frac{\pi}{2}, 0)$ is a saddle point.
- For the critical point $(\frac{3\pi}{2}, 0)$,

  $$D(\frac{3\pi}{2}, 0) = -4 < 0,$$

  so $(\frac{3\pi}{2}, 0)$ is a saddle point.
- For the critical point $(0, 1)$, we have

  $$D(0, 1) = 4 > 0, \quad f_{xx}(0, 1) = 2 > 0,$$

  so $(0, 1)$ is a local minimum.
- For the critical point $(0, -1)$, we have

  $$D(0, -1) = -4 < 0,$$

  so $(0, -1)$ is a saddle point.
- For the critical point $(\pi, 1)$,

  $$D(\pi, 1) = -4 < 0,$$

  so $(\pi, 1)$ is a saddle point.
- For the critical point $(\pi, -1)$,

  $$D(\pi, -1) = 4 > 0, \quad f_{xx}(\pi, -1) = 2 > 0,$$

  so $(\pi, -1)$ is a local minimum.
- For the critical point $(2\pi, 1)$,

  $$D(2\pi, 1) = 4 > 0, \quad f_{xx}(2\pi, 1) = 2 > 0,$$

  so $(2\pi, 1)$ is a local minimum.

- For the critical point $(2\pi, -1)$,
$$D(2\pi, -1) = -4 < 0,$$
so $(2\pi, -1)$ is a saddle point.

(8) We first find the critical points. Note that
$$f_x(x,y) = -2(x^2-1)\cdot(2x) - 2(x^2y-x-1)\cdot(2xy-1) = -4x(x^2-1) - 2(2xy-1)(x^2y-x-1),$$
$$f_y(x,y) = -2(x^2y-x-1)\cdot(x^2) = -2x^2(x^2y-x-1).$$
So if (x,y) is a critical point, this means
$$-4x(x^2-1) - 2(2xy-1)(x^2y-x-1) = 0, \quad -2x^2(x^2y-x-1) = 0.$$
The second requirement means either $x = 0$ or $x^2y - x - 1 = 0$.
- If $x = 0$, then the first requirement becomes
$$-2(-1)(-1) = 0,$$
which is absurd.
- If $x^2y - x - 1 = 0$, then the first requirement becomes
$$-4x(x^2-1) = 0,$$
so either $x = 0$, $x = 1$ or $x = -1$.
  - If $x = 0$, then $x^2y - x - 1 = 0$ becomes $-1 = 0$, which is absurd.
  - If $x = 1$, then $x^2y - x - 1 = 0$ becomes $y - 2 = 0$, or $y = 2$. So $(1,2)$ is a critical point.
  - If $x = -1$, then $x^2y - x - 1 = 0$ becomes $y = 0$, so $(-1,0)$ is a critical point.

So the critical points are $(1,2)$ and $(-1,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that
$$\begin{aligned} f_{xx}(x,y) &= -4(x^2-1) - 4x\cdot(2x) - 2(2y)(x^2y-x-1) - 2(2xy-1)(2xy-1) \\ &= -4(x^2-1) - 8x^2 - 4y(x^2y-x-1) - 2(2xy-1)^2 \\ f_{xy}(x,y) &= -2(2x)(x^2y-x-1) - 2(2xy-1)x^2 \\ f_{yy}(x,y) &= -2x^4. \end{aligned}$$
Note that for both $(x,y) = (1,2)$ and $(-1,0)$, we had $x^2y - x - 1 = 0$ and $x^2 = 1$. Using this, we have
$$f_{xx}(1,2) = -8 - 2(4-1)^2 = -8 - 18 = -26,$$
$$f_{xy}(1,2) = -2(4-1) = -6,$$
$$f_{yy}(1,2) = -2,$$
so $D(1,2) = 52 - 36 > 0$, and $f_{xx}(1,2) < 0$, so $(1,2)$ is a local maximum. For $(-1,0)$, we have
$$f_{xx}(-1,0) = -8 - 2(-1)^2 = -10,$$
$$f_{xy}(-1,0) = -2(-1) = 2,$$
$$f_{yy}(-1,0) = -2,$$
so $D(-1,0) = 20 - 4 > 0$, and $f_{xx}(-1,0) < 0$, so $(-1,0)$ is a local maximum.

(9) We first find the critical points. Note that

$$f_x(x,y) = 3e^y - 3x^2, \quad f_y(x,y) = 3xe^y - 3e^{3y},$$

so if (x,y) is a critical point, it means

$$3e^y - 3x^2 = 0, \quad 3xe^y - 3e^{3y} = 0.$$

The second requirement says $3xe^y = 3e^{3y}$, or $x = e^{2y}$. The first requirement says $3e^y = 3x^2$, or $e^y = x^2$. Thus,

$$x = e^{2y} = (e^y)^2 = (x^2)^2 = x^4.$$

This means either $x = 0$ or $x^3 = 1$, or $x = 1$. If $x = 0$, then $e^{2y} = 0$, which is absurd. If $x = 1$, then $e^{2y} = 1$, so $y = 0$. Thus there is only one critical point, $(1,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = -6x, \quad f_{xy}(x,y) = 3e^y, \quad f_{yy}(x,y) = 3xe^y - 9e^{3y}.$$

So

$$f_{xx}(1,0) = -6, \quad f_{xy}(1,0) = 3, \quad f_{yy}(1,0) = -6.$$

So

$$D(1,0) = 36 - 9 > 0, \quad f_{xx}(1,0) = -6 < 0,$$

so $(1,0)$ is a local maximum.

□

17. Global maxima and minima I

Exercise 1. Find the global maximum and minimum values of f on the given domain.

(1) $f(x,y) = x^2 - y^2$, on the domain $x^2 + y^2 = 1$
(2) $f(x,y) = xe^y$, on the domain $x^2 + y^2 = 2$
(3) $f(x,y) = xye^{-x^2-y^2}$, on the domain $x^2 + y^2 = 1$

Solution.

(1) The domain equation is $g(x,y) = 1$ where $g(x,y) = x^2 + y^2$. So the global max/min can occur at the Lagrange critical points, namely when $\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel. Since $\nabla f(x,y) = \langle 2x, -2y\rangle$ and $\nabla g(x,y) = \langle 2x, 2y\rangle$, this happens either when $\nabla g(x,y) = \langle 0,0\rangle$, which is when $x = y = 0$, which contradicts $x^2 + y^2 = 1$, or there is λ such that $\langle 2x, -2y\rangle = \lambda\langle 2x, 2y\rangle$. Since $2x = 2\lambda x$, either $\lambda = 1$ or $x = 0$.
- If $\lambda = 1$, then $-2y = 2y$, so $y = 0$. Then $x^2 = 1$, so $f(x,y) = 1$.
- If $x = 0$, then $y^2 = 1$, so $f(x,y) = -1$.

So the global max is 1 and the global min is -1.

(2) The domain equation is $g(x,y) = 2$ where $g(x,y) = x^2 + y^2$. So the global max/min can occur at the Lagrange critical points, namely when $\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel. Since $\nabla f(x,y) = \langle e^y, xe^y\rangle$ and $\nabla g(x,y) = \langle 2x, 2y\rangle$, This happens either when $\nabla g(x,y) = \langle 0,0\rangle$, which is when $x = y = 0$, which contradicts $x^2 + y^2 = 2$, or there is λ such that $\langle e^y, xe^y\rangle = \lambda\langle 2x, 2y\rangle$. So $2\lambda x^2 = 2\lambda y$, so either $\lambda = 0$ or $x^2 = y$.
- If $\lambda = 0$, then $e^y = 0$, which is a contradiction.
- If $x^2 = y$, then $x^2 + y^2 = 2$ becomes $x^4 + x^2 - 2 = 0$. This factorizes into $(x^2 - 1)(x^2 + 2) = 0$. So either $x^2 = 1$ or $x^2 = -2$. Since x^2 is positive, $x^2 = 1$, so either $x = 1$ or $x = -1$. Thus $y = 1$.

So the Lagrange critical points are $(1,1)$ and $(-1,1)$. Since $f(1,1) = e$ and $f(-1,1) = -e$, the global max is e and the global min is $-e$.

(3) The domain equation is $g(x,y) = 1$ where $g(x,y) = x^2 + y^2$. So the global max/min can occur at the Lagrange critical points, namely when $\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel. Since $\nabla f(x,y) = \langle ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2}, xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2}\rangle$ and $\nabla g(x,y) = \langle 2x, 2y\rangle$, they can be parallel if there is λ such that

$$(1-2x^2)ye^{-x^2-y^2} = 2\lambda x, \quad (1-2y^2)xe^{-x^2-y^2} = 2\lambda y.$$

So

$$(1-2x^2)y^2e^{-x^2-y^2} = 2\lambda xy = (1-2y^2)x^2e^{-x^2-y^2}$$

so

$$(1-2x^2)y^2 = (1-2y^2)x^2$$

Expanding out we get

$$y^2 - 2x^2y^2 = x^2 - 2x^2y^2$$

or $y^2 = x^2$. So either $x = y$ or $x = -y$. Using $x^2 + y^2 = 1$, we get $2x^2 = 1$, or $x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$. The function $x^2 + y^2$ is set to be 1, and xy is maximized when $x = y$ which is $1/2$ and minimized when $x = -y$ which is $-1/2$. So the global maximum is $\frac{e^{-1}}{2}$ and the global minimum is $-\frac{e^{-1}}{2}$.

□

Exercise 2. Find the global maximum and minimum values of f on the given domain.

(1) $f(x,y) = x^2 + y^2 + 4x - 4y$, on $x^2 + y^2 \leq 9$
(2) $f(x,y) = \sin(x+y)$, on $x^2 + xy + y^2 \leq 3$

Solution. Keep in mind that, in this case, we need to look for

- critical points on the domain, and
- Lagrange critical points on the boundary of the domain.

(1)
- Critical points on the domain.
  Note that
  $$\nabla f(x,y) = \langle 2x+4, 2y-4\rangle$$
  so $\nabla f(x,y) = \langle 0,0\rangle$ means $2x+4 = 0$ and $2y - 4 = 0$, or $x = -2$ and $y = 2$. Since $(-2,2)$ does belong to the domain $x^2 + y^2 \leq 9$, $(-2,2)$ is a critical point.
- Lagrange critical points on the boundary of the domain.
  On the boundary we have a domain equation $g(x,y) = 9$, where $g(x,y) = x^2 + y^2$. By the method of Lagrange multipliers, we would like to find a point (x,y) where $\nabla f(x,y) = \langle 2x+4, 2y-4$ is parallel to $\nabla g(x,y) = \langle 2x, 2y\rangle$. This can happen either when $\nabla g(x,y) = \langle 0,0\rangle$ or there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.
  - If $\nabla g(x,y) = \langle 0,0\rangle$, this means $x = y = 0$, which does not satisfy the domain equation $x^2 + y^2 = 9$.
  - Suppose there is λ such that $\langle 2x+4, 2y-4\rangle = \lambda\langle 2x, 2y\rangle$. Then
    $$2x + 4 = 2\lambda x, \quad 2y - 4 = 2\lambda y$$
    or
    $$2 = (\lambda - 1)x, \quad -2 = (\lambda - 1)y$$
    so $(\lambda - 1)x = -(\lambda - 1)y$. Thus, either $\lambda = 1$ or $x = -y$.

    * If $\lambda = 1$, then $2 = (\lambda - 1)x$ implies $2 = 0$, so this doesn't make sense.
    * If $x = -y$, then $x^2 + y^2 = 9$ implies that $2x^2 = 9$, or $x = \frac{3}{\sqrt{2}}$ or $-\frac{3}{\sqrt{2}}$. Thus the Lagrange critical points are $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ and $(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$.

  Thus the Lagrange critical points are $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ and $(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$.

The job is to compare the values of $f(x, y)$ on the points we've found.

- Critical points on the domain.
  - $f(-2, 2) = -8$.
- Lagrange critical points on the boundary of the domain.
  - $f(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}) = 9 + \frac{12}{\sqrt{2}} + \frac{12}{\sqrt{2}} = 9 + 12\sqrt{2}$.
  - $f(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}) = 9 - \frac{12}{\sqrt{2}} - \frac{12}{\sqrt{2}} = 9 - 12\sqrt{2}$.

Among these values, the largest value $9 + 12\sqrt{2}$ is the global maximum value, and the smallest value -8 is the global minimum value.

(2)
- Critical points on the domain.
  We have

$$\nabla f(x, y) = \langle \cos(x + y), \cos(x + y) \rangle$$

  so the critical points happen when $\cos(x + y) = 0$. This happens when $x + y$ is an odd integer times $\frac{\pi}{2}$ (such as $\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}$). Thus critical points are (x, y) when $x + y$ is an odd integer times $\frac{\pi}{2}$.
- Lagrange critical points on the boundary of the domain.
  On the boundary we have a domain equation $g(x, y) = 3$ where $g(x, y) = x^2 + xy + y^2$. Lagrange critical points are when $\nabla f(x, y) = \langle \cos(x + y), \cos(x + y) \rangle$ is parallel to $\nabla g(x, y) = \langle 2x + y, x + 2y \rangle$. This happens when either $\nabla g(x, y)$ is zero or there is a number λ such that $\nabla f(x, y) = \lambda \nabla g(x, y)$.
  - If $\nabla g(x, y) = \langle 0, 0 \rangle$, this means $2x + y = 0$ and $x + 2y = 0$. Subtracting the second equation from the first equation, we get $x - y = 0$, or $x = y$. Plugging this back into $2x + y = 0$, we get $3x = 0$, or $x = 0$. So $x = y = 0$. This conflicts with the domain equation $x^2 + xy + y^2 = 3$.
  - If there is a number λ such that $\langle \cos(x + y), \cos(x + y) \rangle = \lambda \langle 2x + y, x + 2y \rangle$, this means $\lambda(2x + y) = \lambda(x + 2y)$, or $\lambda(x - y) = 0$. So either $\lambda = 0$ or $x = y$.
    * If $\lambda = 0$, then $\cos(x+y) = 0$, so $\nabla f(x, y) = \langle 0, 0 \rangle$, so this case is subsumed by the critical points on the domain.
    * If $x = y$, then the domain equation $x^2 + xy + y^2 = 3$ becomes $3x^2 = 3$, or $x^2 = 1$, so $x = 1$ or $x = -1$.

  Thus the Lagrange critical points are $(1, 1)$, $(-1, -1)$ and possibly some critical points (namely, $x + y$ is an odd integer times $\frac{\pi}{2}$).

The job is to compare the values of $f(x, y)$ on the points we've found.

- Critical points on the domain.
  - When $x + y$ is an odd integer times $\frac{\pi}{2}$, $\sin(x + y) = 1$ or -1.
- Lagrange critical points on the boundary of the domain.
  - $f(1, 1) = \sin(2)$.
  - $f(-1, -1) = -\sin(2)$.
  - Additionally, there might be some points with $x + y$ equal to an odd integer times $\frac{\pi}{2}$, but these values were already considered above.

Among these values, the largest value 1 is the global maximum value, and the smallest value -1 is the global minimum value.

$\square$

18. Global maxima and minima II

Exercise 1. Find the global maximum and minimum values of f on the given domain.

(1) $f(x,y,z) = xy^2z$, on the domain $\{x^2+y^2+z^2=4\}$
(2) $f(x,y,z) = x^2+y^2+z^2$, on the domain $\{x^2+y^2+z^2+xy=12\}$
(3) $f(x,y,z) = x^4+y^4+z^4$, on the domain $\{x^2+y^2+z^2=1\}$

Solution. The problems in this Exercise are in the case of 3 variables and 1 equality. In this case, we know there is no boundary (the domain has no inequalities in its expression), and we only look for Lagrange critical points of the given domain.

(1) The domain is $g(x,y,z)=4$ where $g(x,y,z)=x^2+y^2+z^2$.

$$\nabla f(x,y,z) = \langle y^2z, 2xyz, xy^2\rangle, \quad \nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$$

In order for them to be parallel, either ∇g is zero or there is λ such that $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$.

- If $\nabla g(x,y,z) = \langle 0,0,0\rangle$, then $x=y=z=0$, which cannot happen as $x^2+y^2+z^2 = 4$.
- If there is λ such that $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$, then

$$y^2z = 2\lambda x, \quad 2xyz = 2\lambda y, \quad xy^2 = 2\lambda z$$

So

$$xy^2z = 2\lambda x^2, \quad xy^2z = \lambda y^2, \quad xy^2z = 2\lambda z^2$$

so

$$2\lambda x^2 = \lambda y^2 = 2\lambda z^2$$

so either $\lambda = 0$ or $2x^2 = y^2 = 2z^2$.

  - If $\lambda=0$, then the equations are $y^2z=0$, $2xyz=0$ and $xy^2=0$. From $y^2z=0$, we see that either $y=0$ or $z=0$.
    * If $y=0$, all three equations are satisfied, and the only remaining condition to check is $x^2+z^2=4$.
    * If $z=0$, then the third equation $xy^2=0$ implies that either $x=0$ or $y=0$. As $y=0$ case is already seen above, we can exclude it and say $x=0$. Then $x=z=0$ with $x^2+y^2+z^2=4$ implies that $y^2=4$. Thus, $y=2$ or $y=-2$.

    Thus, in this case, the Lagrange critical points we obtain are either of the form $(x,0,z)$ with $x^2+z^2=4$, or $(0,2,0)$, or $(0,-2,0)$.
  - If $2x^2=y^2=2z^2$, we use this with $x^2+y^2=z^2=4$ to get

$$x^2+2x^2+x^2=4,$$

    or $x^2=1$. Thus either $x=1$ or $x=-1$. Similarly, as $y^2=2$, and $z^2=1$, so either $y=\sqrt{2}$ or $y=-\sqrt{2}$, and either $z=1$ or $z=-1$.
    Thus, in this case, the Lagrange critical points we obtain are either $(1,\sqrt{2},1)$, $(1,\sqrt{2},-1)$, $(1,-\sqrt{2},1)$, $(1,-\sqrt{2},-1)$, $(-1,\sqrt{2},1)$, $(-1,\sqrt{2},-1)$, $(-1,-\sqrt{2},1)$, $(-1,-\sqrt{2},-1)$.

The values of $f(x,y,z) = xy^2z$ at the points we found are:

- At $(x,0,z)$ with $x^2+z^2=4$, $f(x,0,z)=0$
- $f(0,2,0)=0$
- $f(0,-2,0)=0$
- $f(1,\sqrt{2},1)=2$
- $f(1,\sqrt{2},-1)=-2$
- $f(1,-\sqrt{2},1)=2$
- $f(1,-\sqrt{2},-1)=-2$
- $f(-1,\sqrt{2},1)=-2$
- $f(-1,\sqrt{2},-1)=2$
- $f(-1,-\sqrt{2},1)=-2$
- $f(-1,-\sqrt{2},-1)=2$

Combining all these, the global maximum value is 2 and the global minimum value is -2.

(2) The domain is $g(x,y,z)=12$ where $g(x,y,z)=x^2+y^2+z^2+xy$.

$$\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle, \quad \nabla g(x,y,z) = \langle 2x+y, 2y+x, 2z\rangle$$

For them to be parallel, either $\nabla g(x,y,z)$ is zero or there is λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.

- If $\nabla g(x,y,z) = \langle 0,0,0\rangle$, then $z=0$, and $2x+y=0$ and $2y+x=0$. This solves into $x-y=0$, so $x=y$, so $x=y=0$. This in turn is impossible as $x^2+y^2+z^2+xy=12$.
- If there is λ such that $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$, we have

  $$2x = \lambda(2x+y), \quad 2y = \lambda(2y+x), \quad 2z = 2\lambda z$$

  From the third equation, either $\lambda=1$ or $z=0$.

  – If $\lambda = 1$, we have

    $$2x = 2x+y, \quad 2y = 2y+x$$

    so $x=0$ and $y=0$. The domain equation then becomes $z^2=12$. Thus, either $z=\sqrt{12}$ or $z=-\sqrt{12}$.

    Thus, the Lagrange critical points we obtain in this case are $(0,0,\sqrt{12})$ and $(0,0,-\sqrt{12})$.

  – If $z=0$, we have

    $$2x = \lambda(2x+y), \quad 2y = \lambda(2y+x)$$

    Adding these two, we get

    $$2(x+y) = 3\lambda(x+y)$$

    so either $x+y=0$ or $2=3\lambda$.

    ∗ If $x+y=0$, or $y=-x$, we get $2x=\lambda x$, so either $\lambda=0$ or $x=0$.

      · If $\lambda=0$, then this means $x=y=z=0$, which is not allowed.

      · If $x=0$, then $y=0$, and we already had $z=0$, so $x=y=z=0$ which is not allowed.

    ∗ If $\lambda=\frac{2}{3}$, then $2x=\frac{2}{3}(2x+y)$ implies $6x=4x+2y$, or $2x=2y$, or $x=y$. Putting this into the domain equation, we get $3x^2=12$, or $x^2=4$. So either $x=2$ or $x=-2$. As $x=y$, the Lagrange critical points we obtain in this case are $(2,2,0)$ and $(-2,-2,0)$.

The values of $f(x,y,z) = x^2 + y^2 + z^2$ at the point we found are:

- $f(0,0,\sqrt{12}) = 12$
- $f(0,0,-\sqrt{12}) = 12$
- $f(2,2,0) = 8$
- $f(-2,-2,0) = 8$

So the global maximum value is 12, and the global minimum value is 8.

(3) The domain equation is $g(x,y,z) = 1$ where $g(x,y,z) = x^2 + y^2 + z^2$.

$$\nabla f(x,y,z) = \langle 4x^3, 4y^3, 4z^3 \rangle, \quad \nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$$

For them to be parallel, either $\nabla g(x,y,z)$ is zero or there is λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.

- If $\nabla g(x,y,z) = \langle 0,0,0 \rangle$, $x = y = z = 0$, which is not allowed as $x^2 + y^2 + z^2 = 1$.
- If there is λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, then

$$4x^3 = 2\lambda x, \quad 4y^3 = 2\lambda y, \quad 4z^3 = 2\lambda z$$

  So from the first equation, either $x = 0$ or $2x^2 = \lambda$.
  - If $x = 0$, the second equation says either $y = 0$ or $2y^2 = \lambda$.
    * If $y = 0$, then $x = y = 0$, so $z^2 = 1$. Thus, either $z = 1$ or $z = -1$.
      Thus the Lagrange critical points we obtain in this case are $(0,0,1)$ and $(0,0,-1)$.
    * If $2y^2 = \lambda$, the third equation says either $z = 0$ or $2z^2 = \lambda$.
      · If $z = 0$, then $x = z = 0$, so $y^2 = 1$. Thus, either $y = 1$ or $y = -1$.
      Thus the Lagrange critical points we obtain in this case are $(0,1,0)$ and $(0,-1,0)$.
      · If $2z^2 = \lambda$, then $2y^2 = 2z^2$. As $x = 0$, $x^2 + y^2 + z^2 = 1$ implies that $y^2 = z^2 = \frac{1}{2}$. This implies that either $y = \frac{1}{\sqrt{2}}$ or $y = -\frac{1}{\sqrt{2}}$, and either $z = \frac{1}{\sqrt{2}}$ or $z = -\frac{1}{\sqrt{2}}$.
      Thus the Lagrange critical points we obtain in this case are $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
  - If $2x^2 = \lambda$, the second equation says either $y = 0$ or $2y^2 = \lambda$.
    * If $y = 0$, then the third equation says either $z = 0$ or $2z^2 = \lambda$.
      · If $z = 0$, then $y = z = 0$, so $x^2 = 1$. Thus, either $x = 1$ or $x = -1$.
      Thus the Lagrange critical points we obtain in this case are $(1,0,0)$ and $(-1,0,0)$.
      · If $2z^2 = \lambda$, then $2x^2 = 2z^2$. From $x^2 + y^2 + z^2 = 1$ and $y = 0$, we get $x^2 = z^2 = \frac{1}{2}$. Thus, either $x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$, and either $z = \frac{1}{\sqrt{2}}$ or $z = -\frac{1}{\sqrt{2}}$.
      Thus the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$.
    * If $2y^2 = \lambda$, the third equation says either $z = 0$ or $2z^2 = \lambda$.
      · If $z = 0$, then $2x^2 = 2y^2$. From $x^2 + y^2 + z^2 = 1$ and $z = 0$, we get $x^2 = y^2 = \frac{1}{2}$. Thus, either $x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$, and either $y = \frac{1}{\sqrt{2}}$ or $y = -\frac{1}{\sqrt{2}}$.

Thus the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$.

- If $2z^2 = \lambda$, then $2x^2 = 2y^2 = 2z^2$, so from $x^2 + y^2 + z^2 = 1$, we obtain $x^2 = y^2 = z^2 = \frac{1}{3}$. Thus, either $x = \frac{1}{\sqrt{3}}$ or $x = -\frac{1}{\sqrt{3}}$, either $y = \frac{1}{\sqrt{3}}$ or $y = -\frac{1}{\sqrt{3}}$, and either $z = \frac{1}{\sqrt{3}}$ or $z = -\frac{1}{\sqrt{3}}$.

  Thus the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

The values of $f(x, y, z) = x^4 + y^4 + z^4$ at the points we found are:

- $f(0, 0, 1) = 1$
- $f(0, 0, -1) = 1$
- $f(0, 1, 0) = 1$
- $f(0, -1, 0) = 1$
- $f(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(1, 0, 0) = 1$
- $f(-1, 0, 0) = 1$
- $f(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$
- $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \frac{1}{2}$
- $f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) = \frac{1}{2}$
- $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \frac{1}{2}$
- $f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) = \frac{1}{2}$
- $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3}$
- $f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3}$

(Really you don't have to write all down like this because we know $f(x, y, z)$ does not care about the sign of x, y, z, so you could for example express the last 8 rows compactly as $f(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}) = \frac{1}{3}$; I am just writing like this for completeness)

Thus, the global minimum value is $\frac{1}{3}$, and the global maximum value is 1.

$\square$

Exercise 2. Find the global maximum and minimum values of f on the given domain.

(1) $f(x, y, z) = xyz$, on the domain $\{x^2 + y^2 + z^2 \leq 1\}$
(2) $f(x, y, z) = x^2 + y^2 + z^2$, on the domain $\{x^4 + y^4 + z^4 \leq 1\}$
(3) $f(x, y, z) = x^2 + y^2 + z^2$, on the domain $\{x^2 + y^2 + z^2 + xy - xz - yz \leq 1\}$

Solution. The problems in this Exercise are in the case of 3 variables and 1 inequality. In this case, we know we need to look for two types of points:

- critical points of the original domain,
- Lagrange critical points of the boundary domain.

(1)
- Critical points of the original domain.
  Note
  $$\nabla f(x, y, z) = \langle yz, xz, xy \rangle,$$
  so this is zero if $yz = 0$, $xz = 0$, and $xy = 0$. These three equations imply that at least two of x, y, z are zero.
  Thus, the Lagrange critical points we obtain in this case are $(x, 0, 0)$ with $x^2 \leq 1$, $(0, y, 0)$ with $y^2 \leq 1$, and $(0, 0, z)$ with $z^2 \leq 1$.
- Lagrange critical points of the boundary domain.
  The boundary is expressed as $g(x, y, z) = 1$, where $g(x, y, z) = x^2+y^2+z^2$. Lagrange critical points happen when $\nabla f(x, y, z) = \langle yz, xz, xy \rangle$ are parallel to $\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$. This happens either when $\nabla g(x, y, z)$ is zero or there is a number λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
  - If $\langle 2x, 2y, 2z \rangle = \langle 0, 0, 0 \rangle$, $x = y = z = 0$. This contradicts with the domain equation $x^2 + y^2 + z^2 = 1$.
  - If there is a number λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, we have
    $$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z$$
    So
    $$xyz = 2\lambda x^2, \quad xyz = 2\lambda y^2, \quad xyz = 2\lambda z^2$$
    so $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$. So either $\lambda = 0$ or $x^2 = y^2 = z^2$.
    * If $\lambda = 0$, then $\nabla f(x, y, z) = \langle 0, 0, 0 \rangle$, so this is already deal with as a critical point of the original domain.
    * If $x^2 = y^2 = z^2$, then from $x^2 + y^2 + z^2 = 1$, we have $x^2 = y^2 = z^2 = \frac{1}{3}$. Thus, each x, y, z is either $\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$.
      Thus, the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

The values of $f(x, y, z) = xyz$ at the points we found are:
- At $(x, 0, 0)$ with $x^2 \leq 1$, $f(x, 0, 0) = 0$.
- At $(0, y, 0)$ with $y^2 \leq 1$, $f(0, y, 0) = 0$.
- At $(0, 0, z)$ with $z^2 \leq 1$, $f(0, 0, z) = 0$.
- $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3\sqrt{3}}$
- $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = -\frac{1}{3\sqrt{3}}$
- $f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = -\frac{1}{3\sqrt{3}}$

- $f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3\sqrt{3}}$
- $f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = -\frac{1}{3\sqrt{3}}$
- $f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3\sqrt{3}}$
- $f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3\sqrt{3}}$
- $f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = -\frac{1}{3\sqrt{3}}$

From this, the global maximum value is $\frac{1}{3\sqrt{3}}$ and the global minimum value is $-\frac{1}{3\sqrt{3}}$.

(2)
- Critical points of the original domain.
  Note
  $$\nabla f(x, y, z) = \langle 2x, 2y, 2z\rangle$$
  So the critical point is $(0, 0, 0)$.
- Lagrange critical points on the boundary.
  The boundary domain is expressed as $g(x, y, z) = 1$ where $g(x, y, z) = x^4 + y^4 + z^4$. Lagrange critical points happen when $\nabla f(x, y, z)$ is parallel to $\nabla g(x, y, z) = \langle 4x^3, 4y^3, 4z^3\rangle$. This happens either when $\nabla g(x, y, z)$ is zero or when there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
  - If $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$, then $x = y = z = 0$, which contradicts the domain equation $x^4 + y^4 + z^4 = 1$.
  - If there is λ such that $\langle 2x, 2y, 2z\rangle = \lambda\langle 4x^3, 4y^3, 4z^3\rangle$, we have
    $$2x = 4\lambda x^3, \quad 2y = 4\lambda y^3, \quad 2z = 4\lambda z^3$$
    From the first equation, either $x = 0$ or $2 = 4\lambda x^2$.
    * If $x = 0$, the second equation tells either $y = 0$ or $2 = 4\lambda y^2$.
      · If $y = 0$, then $x = y = 0$ implies $z^4 = 1$, so $z^2 = 1$. Thus, either $z = 1$ or $z = -1$.
      Thus, the Lagrange critical points we obtain in this case are $(0, 0, 1)$, $(0, 0, -1)$.
      · If $2 = 4\lambda y^2$, then the third equation tells either $z = 0$ or $2 = 4\lambda z^2$.
      ☐ If $z = 0$, then $x = z = 0$ implies $y^4 = 1$, so $y^2 = 1$. Thus, either $y = 1$ or $y = -1$. Thus, the Lagrange critical points we obtain in this case are $(0, 1, 0)$, $(0, -1, 0)$.
      ☐ If $2 = 4\lambda z^2$, then $y^2 = \frac{1}{2\lambda} = z^2$ while $x = 0$, so $2y^4 = 1$ or $y^4 = \frac{1}{2}$, so $y^2 = \frac{1}{\sqrt{2}} = z^2$. Thus, y and z are either $\frac{1}{\sqrt[4]{2}}$ or $-\frac{1}{\sqrt[4]{2}}$. Thus, the Lagrange critical points we obtain in this case are $(0, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}})$, $(0, \frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}})$, $(0, -\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}})$, $(0, -\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}})$.
    * If $2 = 4\lambda x^2$, then $x^2 = \frac{1}{2\lambda}$. The second equation tells either $y = 0$ or $2 = 4\lambda y^2$.
      · If $y = 0$, then the third equation tells either $z = 0$ or $2 = 4\lambda z^2$.
      ☐ If $z = 0$, then $y = z = 0$ implies that $x^4 = 1$, or $x^2 = 1$. Thus, x is either -1 or 1. Thus, the Lagrange critical points we obtain in this case are $(1, 0, 0)$, $(-1, 0, 0)$.
      ☐ If $2 = 4\lambda z^2$, then $z^2 = \frac{1}{2\lambda}$. So $x^2 = z^2$ while $y = 0$, so $2x^4 = 1$, or $x^4 = \frac{1}{2}$, or $x^2 = \frac{1}{\sqrt{2}} = z^2$. Thus, x and z are either $\frac{1}{\sqrt[4]{2}}$ or $-\frac{1}{\sqrt[4]{2}}$.

Thus, the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt[4]{2}}, 0, \frac{1}{\sqrt[4]{2}})$, $(\frac{1}{\sqrt[4]{2}}, 0, -\frac{1}{\sqrt[4]{2}})$, $(-\frac{1}{\sqrt[4]{2}}, 0, \frac{1}{\sqrt[4]{2}})$, $(-\frac{1}{\sqrt[4]{2}}, 0, -\frac{1}{\sqrt[4]{2}})$.

* If $2 = 4\lambda y^2$, then $y^2 = \frac{1}{2\lambda} = x^2$. The third equation tells either $z = 0$ or $2 = 4\lambda z^2$.
  · If $z = 0$, then $2x^4 = 1$, or $x^4 = \frac{1}{2}$, or $x^2 = \frac{1}{\sqrt{2}} = y^2$. Thus, x and y are either $\frac{1}{\sqrt[4]{2}}$ or $-\frac{1}{\sqrt[4]{2}}$.
  Thus, the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$, $(\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}, 0)$, $(-\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$, $(-\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}, 0)$.
  · If $2 = 4\lambda z^2$, then $z^2 = \frac{1}{2\lambda} = x^2 = y^2$, so $3x^4 = 1$, or $x^4 = \frac{1}{3}$, or $x^2 = y^2 = z^2 = \frac{1}{\sqrt{3}}$. Thus, x, y, z are either $\frac{1}{\sqrt[4]{3}}$ or $-\frac{1}{\sqrt[4]{3}}$.
  Thus, the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$, $(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}})$, $(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$, $(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}})$, $(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$, $(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}})$, $(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$, $(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}})$.

The values of $f(x, y, z) = x^2 + y^2 + z^2$ at the points we found are:

- $f(0, 0, 0) = 0$
- $f(0, 0, 1) = 1$
- $f(0, 0, -1) = 1$
- $f(0, 1, 0) = 1$
- $f(0, -1, 0) = 1$
- $f(0, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(0, \frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(0, -\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(0, -\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(1, 0, 0) = 1$
- $f(-1, 0, 0) = 1$
- $f(\frac{1}{\sqrt[4]{2}}, 0, \frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(\frac{1}{\sqrt[4]{2}}, 0, -\frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(-\frac{1}{\sqrt[4]{2}}, 0, \frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(-\frac{1}{\sqrt[4]{2}}, 0, -\frac{1}{\sqrt[4]{2}}) = \sqrt{2}$
- $f(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0) = \sqrt{2}$
- $f(\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}, 0) = \sqrt{2}$
- $f(-\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0) = \sqrt{2}$
- $f(-\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}, 0) = \sqrt{2}$
- $f(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}) = \sqrt{3}$
- $f(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}) = \sqrt{3}$
- $f(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}) = \sqrt{3}$
- $f(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}) = \sqrt{3}$
- $f(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}) = \sqrt{3}$

- $f(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}) = \sqrt{3}$
- $f(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}) = \sqrt{3}$
- $f(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}) = \sqrt{3}$

Thus, the global maximum value is $\sqrt{3}$, and the global minimum value is 0.

(3)
- Critical points of the original domain.
  Note
  $$\nabla f(x, y, z) = \langle 2x, 2y, 2z\rangle$$
  so the critical point is $(0, 0, 0)$.
- Lagrange critical points of the boundary domain.
  The boundary domain is expressed as $g(x, y, z) = 1$ where $g(x, y, z) = x^2 + y^2 + z^2 + xy - xz - yz$. Lagrange critical points happen when $\nabla f(x, y, z)$ is parallel to $\nabla g(x, y, z) = \langle 2x+y-z, 2y+x-z, 2z-x-y\rangle$. This happens when either $\nabla g(x, y, z)$ is zero or there is a number λ such that $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$.
  - If $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$, we have
    $$2x + y - z = 0, \quad 2y + x - z = 0, \quad 2z - x - y = 0$$
    If we add all three, we get $2x + 2y = 0$, or $x + y = 0$. So $2z = 0$, so $z = 0$. So $2x + y = x = 0$, so $x = 0$, and $y = 0$. But $x = y = z = 0$ contradicts the boundary domain equation $x^2 + y^2 + z^2 + xy - xz - yz = 1$.
  - If there is λ such that $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$ we have
    $$2x = \lambda(2x + y - z), \quad 2y = \lambda(2y + x - z), \quad 2z = \lambda(2z - x - y)$$
    If you subtract the second equation from the first equation, you get
    $$2(x - y) = \lambda(x - y),$$
    so either $x - y = 0$ or $\lambda = 2$.
    * If $x - y = 0$, then $x = y$, so we have
      $$2x = \lambda(3x - z), \quad z = \lambda(z - x)$$
      If you add them you get
      $$2x + z = 2\lambda x$$
      so $z = 2(\lambda - 1)x$. On the other hand, the second equation tells you $\lambda x = (\lambda - 1)z$, so
      $$\lambda x = (\lambda - 1)z = 2(\lambda - 1)^2 x = (2\lambda^2 - 4\lambda + 2)x$$
      so either $x = 0$ or $\lambda = 2\lambda^2 - 4\lambda + 2$.
      · If $x = 0$, then $y = 0$, so the domain equation becomes $z^2 = 1$. Thus, either $z = 1$ or $z = -1$.
        Thus, the Lagrange critical points we obtain in this case are $(0, 0, 1)$, $(0, 0, -1)$.
      · If $\lambda = 2\lambda^2 - 4\lambda + 2$, then $2\lambda^2 - 5\lambda + 2 = 0$, so $(2\lambda - 1)(\lambda - 2) = 0$, so either $\lambda = \frac{1}{2}$ or $\lambda = 2$.
        □ If $\lambda = \frac{1}{2}$, then we have
        $$2x = \frac{1}{2}(3x - z), \quad z = \frac{1}{2}(z - x)$$

or

$$4x = 3x - z, \quad 2z = z - x$$

or $z = -x$. So $x = y = -z$. The domain equation becomes $6x^2 = 1$, or $x^2 = \frac{1}{6}$. Thus either $x = \frac{1}{\sqrt{6}}$ or $x = -\frac{1}{\sqrt{6}}$, and $y = x$ and $z = -x$. Thus, the Lagrange critical points we obtain in this case are $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$, $(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$.

□ The $\lambda = 2$ case is subsumed to the later more general case below.

∗ If $\lambda = 2$, then we have

$$x = 2x + y - z, \quad y = 2y + x - z, \quad z = 2z - x - y,$$

so all three are exactly equivalent to $x + y = z$. The boundary domain equation becomes

$$x^2 + y^2 + (x + y)^2 + xy - (x + y)^2 = 1,$$

or $x^2 + xy + y^2 = 1$.

Thus, the Lagrange critical points we obtain in this case are $(x, y, x + y)$, where $x^2 + xy + y^2 = 1$.

The values of $f(x, y, z) = x^2 + y^2 + z^2$ at the points we found are

- $f(0, 0, 0) = 0$
- $f(0, 0, 1) = 1$
- $f(0, 0, -1) = 1$
- $f(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}) = \frac{1}{2}$
- $f(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) = \frac{1}{2}$
- At $(x, y, x + y)$ with $x^2 + xy + y^2 = 1$, $f(x, y, x + y) = x^2 + y^2 + (x + y)^2 = 2x^2 + 2xy + 2y^2 = 2(x^2 + xy + y^2) = 2$.

Thus, the global maximum value is 2 and the global minimum value is 0.

□

Exercise 3. Find the global maximum and minimum values of f on the given domain.

(1) $f(x, y, z) = z$ on the domain $\{x^2 + y^2 + z^2 = 1,\ x + y - z = 0\}$
(2) $f(x, y, z) = x^2 + y^2$ on the domain $\{x^2 + y^2 + z^2 = 50,\ x - z = 0\}$

Solution. The problems in this Exercise are in the case of 3 variables and 2 equalities. In this case, we know we need to look for the Lagrange critical points of the given domain.

(1) The domain equations are $g(x, y, z) = 1$ and $h(x, y, z) = 0$ where $g(x, y, z) = x^2+y^2+z^2$ and $h(x, y, z) = x + y - z$. Thus

$$\nabla f(x, y, z) = \langle 0, 0, 1\rangle, \quad \nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle, \quad \nabla h(x, y, z) = \langle 1, 1, -1\rangle$$

Lagrange critical points are when either $\nabla g(x, y, z)$ or $\nabla h(x, y, z)$ are zero, or $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)+\mu\nabla h(x, y, z)$. Note that $\nabla h(x, y, z)$ is not zero, and $\nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$ is zero if $x = y = z = 0$, which does not lie in the domain because of the domain equation $x^2 + y^2 + z^2 = 1$. Thus we need to solve

$$\nabla f(x, y, z) = \lambda\nabla g(x, y, z) + \mu\nabla h(x, y, z)$$

or

$$0 = 2\lambda x + \mu, \quad 0 = 2\lambda y + \mu, \quad 1 = 2\lambda z - \mu$$

From the first two equations, $2\lambda x = 2\lambda y$, so either $\lambda = 0$ or $x = y$.

- If $\lambda = 0$, we have $\mu = 0$ and $1 = -\mu$, which is a contradiction.
- If $x = y$, the domain equations say $2x^2+z^2 = 1$ and $2x = z$. Thus, $6x^2 = 1$, or $x = \frac{1}{\sqrt{6}}$ or $-\frac{1}{\sqrt{6}}$. Thus the Lagrange critical points are $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$ and $(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$.

The values of $f(x, y, z)$ at the Lagrange critical points are $f(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}) = \frac{2}{\sqrt{6}}$ and $f(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}) = -\frac{2}{\sqrt{6}}$. Thus, the global maximum value is $\frac{2}{\sqrt{6}}$ and the global minimum value is $-\frac{2}{\sqrt{6}}$.

(2) The domain equations are $g(x, y, z) = 50$ and $h(x, y, z) = 0$ where $g(x, y, z) = x^2 + y^2 + z^2$ and $h(x, y, z) = x - z$. This happens either when $\nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$ or $\nabla h(x, y, z) = \langle 1, 0, -1\rangle$ is zero, or when $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$. The former case can only happen when $2x = 2y = 2z = 0$, which does not satisfy $x^2 + y^2 + z^2 = 50$. Thus we need to solve

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

or

$$\langle 2x, 2y, 0\rangle = \lambda\langle 2x, 2y, 2z\rangle + \mu\langle 1, 0, -1\rangle$$

or

$$2x = 2\lambda x + \mu, \quad 2y = 2\lambda y, \quad 0 = 2\lambda z - \mu$$

From the second equation, either $\lambda = 1$ or $y = 0$.

- If $\lambda = 1$, we have $2x = 2x + \mu$ or $0 = 2z - \mu$. So, $\mu = 0$ and $z = 0$. Then $x - z = 0$ means $x = 0$, so $y^2 = 50$. Thus the Lagrange critical points are $(0, \sqrt{50}, 0)$ and $(0, -\sqrt{50}, 0)$.
- If $y = 0$, then $x^2 + z^2 = 2x^2 = 50$, so the Lagrange critical points are $(5, 0, 5)$ and $(-5, 0, -5)$.

The values at the Lagrange critical points are $f(0, \sqrt{50}, 0) = 50$, $f(0, -\sqrt{50}, 0) = 50$, $f(5, 0, 5) = 25$, $f(-5, 0, -5) = 25$. Thus, the global maximum value is 50 and the global minimum value is 25.

□

19. Global maxima and minima III

Exercise 1. List all the nonempty boundary pieces of the domain. Mark every boundary piece that is a bunch of points.

(1) $\{(x, y) \mid 0 \le x + y \le 1\}$
(2) $\{(x, y) \mid x^2 + 4y^2 \le 4,\ x \ge 1\}$
(3) $\{(x, y) \mid x + 2y^2 \le 0,\ x + y \le -1\}$
(4) $\{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$
(5) $\{(x, y, z) \mid x^2 + y^2 + z^2 \le 1,\ x + y \le 1,\ x \ge \frac{1}{2}\}$
(6) $\{(x, y, z) \mid x^2 + y^2 = z^2,\ x + y \ge 1,\ z \le 5\}$

Solution.

(1) What you naturally get are

$$\{(x, y) \mid x + y = 0,\ x + y \le 1\}$$
$$\{(x, y) \mid 0 \le x + y,\ x + y = 1\}$$
$$\{(x, y) \mid x + y = 0,\ x + y = 1\}$$

The third domain is obviously empty. Also, since the condition $x+y \le 1$ is redundant under the other condition $x+y=0$, and since the condition $0 \le x+y$ is redundant under the other condition $x+y=1$, we can express the nonempty boundary pieces more simply as

$$\{(x,y) \mid x+y=0\}$$
$$\{(x,y) \mid x+y=1\}$$

(2)

$$\{(x,y) \mid x^2+4y^2=4,\ x \ge 1\}$$
$$\{(x,y) \mid x^2+4y^2 \le 4,\ x=1\}$$
$$\boxed{\{(x,y) \mid x^2+4y^2=4,\ x=1\}} \leftarrow \text{Bunch of points}$$

The last one is a bunch of points because it is defined by 2 equalities in a 2-variables xy-plane.

(3)

$$\{(x,y) \mid x+2y^2=0,\ x+y \le -1\}$$
$$\{(x,y) \mid x+2y^2 \le 0,\ x+y=-1\}$$
$$\boxed{\{(x,y) \mid x+2y^2=0,\ x+y=-1\}} \leftarrow \text{Bunch of points}$$

The last one is a bunch of points because it is defined by 2 equalities in a 2-variables xy-plane.

(4) What you naturally get are

$$\{(x,y) \mid 0=x,\ x \le 2,\ 0 \le y \le 2\}$$
$$\{(x,y) \mid 0 \le x,\ x=2,\ 0 \le y \le 2\}$$
$$\{(x,y) \mid 0 \le x \le 2,\ 0=y,\ y \le 2\}$$
$$\{(x,y) \mid 0 \le x \le 2,\ 0 \le y, y=2\}$$
$$\{(x,y) \mid 0=x,\ x=2,\ 0 \le y \le 2\}$$
$$\{(x,y) \mid 0=x,\ x \le 2,\ 0=y,\ y \le 2\}$$
$$\{(x,y) \mid 0=x,\ x \le 2,\ 0 \le y, y=2\}$$
$$\{(x,y) \mid 0 \le x,\ x=2,\ 0=y,\ y \le 2\}$$
$$\{(x,y) \mid 0 \le x,\ x=2,\ 0 \le y,\ y=2\}$$
$$\{(x,y) \mid 0 \le x \le 2,\ 0=y, y=2\}$$

But $y=0$, $y=2$ are conflicting conditions, and similarly $x=0$, $x=2$ are conflicting conditions. Moreover, $x \le 2$ is redundant under the condition $x=0$, etc. So, simplifying it, we are left with only 8 boundary pieces,

$$\{(x,y) \mid x=0,\ 0 \le y \le 2\}$$
$$\{(x,y) \mid x=2,\ 0 \le y \le 2\}$$
$$\{(x,y) \mid 0 \le x \le 2,\ y=0\}$$
$$\{(x,y) \mid 0 \le x \le 2,\ y=2\}$$
$$\boxed{\{(x,y) \mid x=0,\ y=0\}} \leftarrow \text{Bunch of points}$$
$$\boxed{\{(x,y) \mid x=0,\ y=2\}} \leftarrow \text{Bunch of points}$$

$$\boxed{\{(x,y) \mid x=2,\ y=0\}} \leftarrow \text{Bunch of points}$$

$$\boxed{\{(x,y) \mid x=2,\ y=2\}} \leftarrow \text{Bunch of points}$$

The last four domains are bunches of points because they are defined by 2 equalities in a 2-variables xy-plane.

(5)

$$\{(x,y,z) \mid x^2+y^2+z^2=1,\ x+y\le 1,\ x\ge \frac{1}{2}\}$$
$$\{(x,y,z) \mid x^2+y^2+z^2\le 1,\ x+y=1,\ x\ge \frac{1}{2}\}$$
$$\{(x,y,z) \mid x^2+y^2+z^2\le 1,\ x+y\le 1,\ x= \frac{1}{2}\}$$
$$\{(x,y,z) \mid x^2+y^2+z^2=1,\ x+y=1,\ x\ge \frac{1}{2}\}$$
$$\{(x,y,z) \mid x^2+y^2+z^2=1,\ x+y\le 1,\ x= \frac{1}{2}\}$$
$$\{(x,y,z) \mid x^2+y^2+z^2\le 1,\ x+y=1,\ x= \frac{1}{2}\}$$
$$\boxed{\{(x,y,z) \mid x^2+y^2+z^2=1,\ x+y=1,\ x= \frac{1}{2}\}} \leftarrow \text{Bunch of points}$$

The last one is a bunch of points because it is defined by 3 equalities in a 3-variables xyz-space.

(6)

$$\{(x,y,z) \mid x^2+y^2=z^2,\ x+y=1,\ z\le 5\}$$
$$\{(x,y,z) \mid x^2+y^2=z^2,\ x+y\ge 1,\ z=5\}$$
$$\boxed{\{(x,y,z) \mid x^2+y^2=z^2,\ x+y=1,\ z=5\}} \leftarrow \text{Bunch of points}$$

The last one is a bunch of points because it is defined by 3 equalities in a 3-variables xyz-space.

□

Exercise 2. Find the global maximum and minimum values of $f(x,y)$ on the domain D.

(1) $f(x,y)=x^2+y^2-2x$, and D is the triangular domain with vertices $(2,0)$, $(0,2)$ and $(0,-2)$, including boundaries.
(2) $f(x,y)=x+y+xy$, and D is the triangular domain with vertices $(0,0)$, $(0,2)$, and $(4,0)$, including boundaries.
(3) $f(x,y)=x^2+y^2+x^2y+4$, and $D=\{(x,y) \mid -1\le x\le 1,\ -1\le y\le 1\}$.
(4) $f(x,y)=x^2+xy+y^2-6y$, and $D=\{(x,y) \mid -3\le x\le 3,\ 0\le y\le 5\}$.
(5) $f(x,y)=x^2+2y^2-2x-4y+1$, and $D=\{(x,y) \mid 0\le x\le 2, 0\le y\le 3\}$.

Solution. (1) The domain is expressed as

$$\{x\ge 0,\ x+y\le 2,\ y-x\ge -2\}$$

There are 7 types of points you have to look for.

(a) Critical points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = 0,\ x + y \leq 2,\ y - x \geq -2\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{x \geq 0,\ x + y = 2,\ y - x \geq -2\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{x \geq 0,\ x + y \leq 2,\ y - x = -2\}$$

(e) All points of the boundary piece #4,

$$\{x = 0,\ x + y = 2,\ y - x \geq -2\}$$

(f) All points of the boundary piece #5,

$$\{x = 0,\ x + y \leq 2,\ y - x = -2\}$$

(g) All points of the boundary piece #6,

$$\{x \geq 0,\ x + y = 2,\ y - x = -2\}$$

Let $g(x, y) = x$, $h(x, y) = x + y$, $i(x, y) = y - x$.

(a) Critical points of the original domain.
The equations are

$$\nabla f(x, y) = \langle 0, 0\rangle, \quad g(x, y) \geq 0,\ h(x, y) \leq 2,\ i(x, y) \geq -2.$$

As $\nabla f(x, y) = \langle 2x - 2, 2y\rangle$, this is equal to $\langle 0, 0\rangle$ exactly when $x = 1$ and $y = 0$. As $(1, 0)$ satisfies all three inequalities, $x \geq 0$, $x + y \leq 2$, $y - x \geq -2$, we get $\boxed{(1, 0)}$ on the list.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = 0,\ x + y \leq 2,\ y - x \geq -2\}$$

Case A The equations are

$$\nabla g(x, y) = \langle 0, 0\rangle, \quad g(x, y) = 0,\ h(x, y) \leq 2,\ i(x, y) \geq -2.$$

As $\nabla g(x, y) = \langle 1, 0\rangle$, this can never be equal to $\langle 0, 0\rangle$. So there are no points from this case.

Case B The equations are

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) - 0,\ h(x, y) \leq 2,\ i(x, y) \geq -2.$$

We have $\langle 2x - 2, 2y\rangle = \lambda\langle 1, 0\rangle = \lambda\lambda, 0\rangle$, so this means that $2y = 0$, or $y = 0$. As $x = 0$, and $(0, 0)$ satisfies both $x + y \leq 2$ and $y - x \geq -2$, we have $\boxed{(0, 0)}$ on the list.

(c) Lagrange critical points of the boundary piece #2,

$$\{x \geq 0,\ x + y = 2,\ y - x \geq -2\}$$

Case A The equations are

$$\nabla h(x, y) = \langle 0, 0\rangle, \quad g(x, y) \geq 0,\ h(x, y) = 2,\ i(x, y) \geq -2.$$

As $\nabla h(x, y) = \langle 1, 1\rangle$, this is never $\langle 0, 0\rangle$. Thus there are no points from this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad g(x,y) \geq 0,\ h(x,y) = 2,\ i(x,y) \geq -2.$$

We have $\langle 2x-2, 2y\rangle = \lambda\langle 1,1\rangle = \langle \lambda, \lambda\rangle$, this implies that $2x-2 = 2y$. Thus $x-1 = y$. As $x+y=2$, we have $2x-1=2$, or $2x=3$, or $x=\frac{3}{2}$, and $y=\frac{1}{2}$. This satisfies both $x \geq 0$ and $y-x \geq -2$, so we get $\boxed{(\frac{3}{2}, \frac{1}{2})}$ on the list.

(d) Lagrange critical points of the boundary piece #3,

$$\{x \geq 0,\ x+y \leq 2,\ y-x = -2\}$$

Case A The equations are

$$\nabla i(x,y) = \langle 0,0\rangle, \quad g(x,y) \geq 0,\ h(x,y) \leq 2,\ i(x,y) = -2.$$

As $\nabla i(x,y) = \langle -1, 1\rangle$, this is never equal to $\langle 0,0\rangle$. Thus there are no points from this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla i(x,y), \quad g(x,y) \geq 0,\ h(x,y) \leq 2,\ i(x,y) = -2.$$

We have $\langle 2x-2, 2y\rangle = \lambda\langle -1,1\rangle = \langle -\lambda, \lambda\rangle$, so this means $2x-2 = -2y$, or $x-1 = -y$, or $x+y=1$. As $y-x=-2$, we have $2y=-1$, or $y=-\frac{1}{2}$, and $x=\frac{3}{2}$. This satisfies both $x \geq 0$ and $x+y \leq 2$, so we get $\boxed{(\frac{3}{2}, -\frac{1}{2})}$ on the list.

(e) All points of the boundary piece #4,

$$\{x = 0,\ x+y = 2,\ y-x \geq -2\}$$

The equations are just the domain equations, so $x=0$ and $y=2$. This satisfies $y-x \geq -2$. Thus we get a point $\boxed{(0,2)}$ on the list.

(f) All points of the boundary piece #5,

$$\{x = 0,\ x+y \leq 2,\ y-x = -2\}$$

The equations are just the domain equations, so $x=0$ and $y=-2$. This satisfies $x+y \leq 2$. Thus we get a point $\boxed{(0,-2)}$ on the list.

(g) All points of the boundary piece #6,

$$\{x \geq 0,\ x+y = 2,\ y-x = -2\}$$

The equations are just the domain equations. Adding $x+y=2$ and $y-x=-2$, we get $2y=0$, or $y=0$. From this, $x=2$. As $x \geq 2$ is satisfied, we get a point $\boxed{(2,0)}$ on the list.

The list of values on the candidate points is:

- $f(1,0) = -1$
- $f(0,0) = 0$
- $f(\frac{3}{2}, \frac{1}{2}) = -\frac{1}{2}$
- $f(\frac{3}{2}, -\frac{1}{2}) = -\frac{1}{2}$
- $f(0,2) = 4$
- $f(0,-2) = 4$
- $f(2,0) = 0$

So the global maximum value is 4, and the global minimum value is -1.

(2) The domain is expressed as

$$\{x \geq 0,\ y \geq 0,\ x+2y \leq 4\}$$

There are 7 types of points you are looking for.

(a) Critical points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = 0,\ y \geq 0,\ x+2y \leq 4\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{x \geq 0,\ y = 0,\ x+2y \leq 4\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{x \geq 0,\ y \geq 0,\ x+2y = 4\}$$

(e) All points of the boundary piece #4,

$$\{x = 0,\ y = 0,\ x+2y \leq 4\}$$

(f) All points of the boundary piece #5,

$$\{x = 0,\ y \geq 0,\ x+2y = 4\}$$

(g) All points of the boundary piece #6,

$$\{x \geq 0,\ y = 0,\ x+2y = 4\}$$

Let $g(x,y) = x$, $h(x,y) = y$, $i(x,y) = x+2y$.

(a) Critical points of the original domain.
The equations are

$$\nabla f(x,y) = \langle 0,0\rangle, \quad g(x,y) \geq 0,\ h(x,y) \geq 0,\ i(x,y) \leq 4$$

As $\nabla f(x,y) = \langle 1+y, 1+x\rangle$, the equation $\nabla f(x,y) = \langle 0,0\rangle$ means $x = y = -1$. This is not on the domain, so we get no points from this case.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = 0,\ y \geq 0,\ x+2y \leq 4\}$$

Case A The equations are

$$\nabla g(x,y) = \langle 0,0\rangle, \quad g(x,y) = 0,\ h(x,y) \geq 0,\ i(x,y) \leq 4$$

As $\nabla g(x,y) = \langle 1,0\rangle$ is never $\langle 0,0\rangle$, we get no points from this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 0,\ h(x,y) \geq 0,\ i(x,y) \leq 4$$

We have $\langle 1+y, 1+x\rangle = \lambda\langle 1,0\rangle = \langle \lambda, 0\rangle$. This implies that $x = -1$. As $x = -1$ is not in the domain, we get no points from this case.

(c) Lagrange critical points of the boundary piece #2,

$$\{x \geq 0,\ y = 0,\ x+2y \leq 4\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle, \quad g(x,y) \geq 0,\ h(x,y) = 0,\ i(x,y) \leq 4$$

As $\nabla h(x,y) = \langle 0,1\rangle$ is never $\langle 0,0\rangle$, we get no points from this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad g(x,y) \geq 0,\ h(x,y) = 0,\ i(x,y) \leq 4$$

We have $\langle 1+y, 1+x\rangle = \lambda\langle 0,1\rangle = \langle 0,\lambda\rangle$, which means $y = -1$. As $y = -1$ is not in the domain, we get no points from this case.

(d) Lagrange critical points of the boundary piece #3,

$$\{x \geq 0,\ y \geq 0,\ x + 2y = 4\}$$

Case A The equations are

$$\nabla i(x,y) = \langle 0,0\rangle, \quad g(x,y) \geq 0,\ h(x,y) \geq 0,\ i(x,y) = 4$$

As $\nabla i(x,y) = \langle 1,2\rangle$ is never $\langle 0,0\rangle$, we get no points from this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla i(x,y), \quad g(x,y) \geq 0,\ h(x,y) \geq 0,\ i(x,y) = 4$$

We have $\langle 1+y, 1+x\rangle = \lambda\langle 1,2\rangle = \langle \lambda, 2\lambda\rangle$, we have $2(1+y) = 1+x$, or $1+2y = x$. As we also have $x + 2y = 4$, this means that $4y + 1 = 4$, or $y = \frac{3}{4}$, and $x = 1 + \frac{3}{2} = \frac{5}{2}$. As this is not in the domain, we get no points from this case.

(e) All points of the boundary piece #4,

$$\{x = 0,\ y = 0,\ x + 2y \leq 4\}$$

This is just the point $\boxed{(0,0)}$.

(f) All points of the boundary piece #5,

$$\{x = 0,\ y \geq 0,\ x + 2y = 4\}$$

From $x = 0$, $2y = 4$, so $y = 2$, so this is the point $\boxed{(0,2)}$.

(g) All points of the boundary piece #6,

$$\{x \geq 0,\ y = 0,\ x + 2y = 4\}$$

From $y = 0$, $x = 4$, so this is the point $\boxed{(4,0)}$.

The list of values on the candidate points is:

- $f(0,0) = 0$
- $f(0,2) = 2$
- $f(4,0) = 4$

So, the global maximum value is 4, and the global minimum value is 0.

(3) There are 9 types of points we need to look for.

(a) Critcial points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = -1,\ -1 \leq y \leq 1\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{x = 1,\ -1 \leq y \leq 1\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{-1 \leq x \leq 1,\ y = -1\}$$

(e) Lagrange critical points of the boundary piece #4,

$$\{-1 \leq x \leq 1,\ y = 1\}$$

(f) All points of the boundary piece #5,

$$\{x = -1,\ y = -1\}$$

(g) All points of the boundary piece #6,

$$\{x = -1,\ y = 1\}$$

(h) All points of the boundary piece #7,

$$\{x = 1,\ y = -1\}$$

(i) All points of the boundary piece #8,

$$\{x = 1,\ y = 1\}$$

Let $g(x, y) = x$, $h(x, y) = y$.

(a) Critcial points of the original domain.
The equations are

$$\nabla f(x, y) = \langle 0, 0\rangle,\ -1 \leq g(x, y) \leq 1,\ -1 \leq h(x, y) \leq 1$$

As $\nabla f(x, y) = \langle 2x + 2xy, 2y + x^2\rangle$, this is equal to $\langle 0, 0\rangle$ means $2x + 2xy = 0$ and $2y + x^2 = 0$. From $2x + 2xy = 2x(1 + y) = 0$, we know either $x = 0$ or $y = -1$.

- If $x = 0$, then $2y + x^2 = 0$ implies $y = 0$. As $(0, 0)$ is in the domain, we get $\boxed{(0, 0)}$ as a point on the list.
- If $y = -1$, then $2y + x^2 = 0$ implies $x^2 = 2$. So either $x = \sqrt{2}$ or $x = -\sqrt{2}$. As we need $-1 \leq x \leq 1$, neither of the options is in the domain, so we get no points in this case.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = -1,\ -1 \leq y \leq 1\}$$

Case A The equations are

$$\nabla g(x, y) = \langle 0, 0\rangle,\ g(x, y) = -1,\ -1 \leq h(x, y) \leq 1$$

As $\nabla g(x, y) = \langle 1, 0\rangle$, this is never $\langle 0, 0\rangle$. Thus we get no points in this case.

Case B The equations are

$$\nabla f(x, y) = \lambda \nabla g(x, y),\ g(x, y) = -1,\ -1 \leq h(x, y) \leq 1$$

This is $\langle 2x + 2xy, 2y + x^2\rangle = \lambda\langle 1, 0\rangle = \langle \lambda, 0\rangle$, so $2y + x^2 = 0$. As $x = -1$, this means $2y = -1$, or $y = -\frac{1}{2}$. As this is in the domain, we get $\boxed{(-1, -\dfrac{1}{2})}$ as a point on the list.

(c) Lagrange critical points of the boundary piece #2,

$$\{x = 1,\ -1 \leq y \leq 1\}$$

Case A The equations are

$$\nabla g(x,y) = \langle 0,0\rangle,\ g(x,y)=1,\ -1\le h(x,y)\le 1$$

As $\nabla g(x,y) = \langle 1,0\rangle$, this is never $\langle 0,0\rangle$. Thus we get no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda\nabla g(x,y),\ g(x,y)=1,\ -1\le h(x,y)\le 1$$

This is $\langle 2x+2xy, 2y+x^2\rangle = \lambda\langle 1,0\rangle = \langle \lambda, 0\rangle$, so $2y+x^2=0$. As $x=1$, this means $2y=-1$, or $y=-\frac{1}{2}$. As this is in the domain, we get $\boxed{(1,-\frac{1}{2})}$ as a point on the list.

(d) Lagrange critical points of the boundary piece #3,

$$\{-1\le x\le 1,\ y=-1\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle,\ -1\le g(x,y)\le 1,\ h(x,y)=-1$$

As $\nabla h(x,y) = \langle 0,1\rangle$, this is never $\langle 0,0\rangle$. Thus we get no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda\nabla h(x,y),\ -1\le g(x,y)\le 1,\ h(x,y)=-1$$

This is $\langle 2x+2xy, 2y+x^2\rangle = \lambda\langle 0,1\rangle = \langle 0, \lambda\rangle$, so $2x+2xy=0$. As $y=-1$, we actually see that any point $\boxed{(x,-1)}$ with $-1\le x\le 1$ is a Lagrange critical point.

(e) Lagrange critical points of the boundary piece #4,

$$\{-1\le x\le 1,\ y=1\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle,\ -1\le g(x,y)\le 1,\ h(x,y)=1$$

As $\nabla h(x,y) = \langle 0,1\rangle$, this is never $\langle 0,0\rangle$. Thus we get no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda\nabla h(x,y),\ -1\le g(x,y)\le 1,\ h(x,y)=1$$

This is $\langle 2x+2xy, 2y+x^2\rangle = \lambda\langle 0,1\rangle = \langle 0, \lambda\rangle$, so $2x+2xy=0$. As $y=1$, we have $4x=0$, so $x=0$. As this is in the domain, we get $\boxed{(0,1)}$ as a point on the list.

(f) All points of the boundary piece #5,

$$\{x=-1,\ y=-1\}$$

This is just the point $\boxed{(-1,-1)}$.

(g) All points of the boundary piece #6,

$$\{x=-1,\ y=1\}$$

This is just the point $\boxed{(-1,1)}$.

(h) All points of the boundary piece #7,

$$\{x = 1,\ y = -1\}$$

This is just the point $\boxed{(1, -1)}$.

(i) All points of the boundary piece #8,

$$\{x = 1,\ y = 1\}$$

This is just the point $\boxed{(1, 1)}$.

The list of values on the candidate points are:

- $f(0, 0) = 4$
- $f(-1, -\frac{1}{2}) = 1 - \frac{1}{4} - \frac{1}{2} + 4 = \frac{17}{4}$
- $f(1, -\frac{1}{2}) = 1 - \frac{1}{4} - \frac{1}{2} + 4 = \frac{17}{4}$
- For $-1 \leq x \leq 1$, $f(x, -1) = x^2 + 1 - x^2 + 4 = 5$
- $f(0, 1) = 5$
- $f(-1, -1) = 5$
- $f(-1, 1) = 7$
- $f(1, -1) = 5$
- $f(1, 1) = 7$

So, the global maximum value is 7, and the global minimum value is 4.

(4) There are 9 types of points we need to look for.

(a) Critcial points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = -3,\ 0 \leq y \leq 5\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{x = 3,\ 0 \leq y \leq 5\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{-3 \leq x \leq 3,\ y = 0\}$$

(e) Lagrange critical points of the boundary piece #4,

$$\{-3 \leq x \leq 3,\ y = 5\}$$

(f) All points of the boundary piece #5,

$$\{x = -3,\ y = 0\}$$

(g) All points of the boundary piece #6,

$$\{x = -3,\ y = 5\}$$

(h) All points of the boundary piece #7,

$$\{x = 3,\ y = 0\}$$

(i) All points of the boundary piece #8,

$$\{x = 3,\ y = 5\}$$

Let $g(x, y) = x$, $h(x, y) = y$.

(a) Critcial points of the original domain.
The equations are

$$\nabla f(x,y) = \langle 0,0\rangle, \quad -3 \le g(x,y) \le 3,\ 0 \le h(x,y) \le 5$$

As $\nabla f(x,y) = \langle 2x+y, 2y-6\rangle$, this being equal to $\langle 0,0\rangle$ means that $2x+y=0$ and $2y-6=0$. In particular, $y=3$ and $2x=-3$, so $x=-\frac{3}{2}$. As this is in the domain, we get a point $\boxed{(-\frac{3}{2},3)}$ on the list.

(b) Lagrange critical points of the boundary piece #1,

$$\{x=-3,\ 0\le y\le 5\}$$

Case A The equations are

$$\nabla g(x,y) = \langle 0,0\rangle, \quad g(x,y) = -3,\ 0 \le h(x,y) \le 5$$

As $\nabla g(x,y) = \langle 1,0\rangle$ is never $\langle 0,0\rangle$, we get no points in this case.

Case B

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = -3,\ 0 \le h(x,y) \le 5$$

We have $\langle 2x+y, 2y-6\rangle = \lambda\langle 1,0\rangle = \langle \lambda, 0\rangle$, so $2y-6=0$, or $y=3$. As $x=-3$, we get $\boxed{(-3,3)}$ on the list.

(c) Lagrange critical points of the boundary piece #2,

$$\{x=3,\ 0\le y\le 5\}$$

Case A The equations are

$$\nabla g(x,y) = \langle 0,0\rangle, \quad g(x,y) = 3,\ 0 \le h(x,y) \le 5$$

As $\nabla g(x,y) = \langle 1,0\rangle$ is never $\langle 0,0\rangle$, we get no points in this case.

Case B

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 3,\ 0 \le h(x,y) \le 5$$

We have $\langle 2x+y, 2y-6\rangle = \lambda\langle 1,0\rangle = \langle \lambda, 0\rangle$, so $2y-6=0$, or $y=3$. As $x=3$, we get $\boxed{(3,3)}$ on the list.

(d) Lagrange critical points of the boundary piece #3,

$$\{-3\le x\le 3,\ y=0\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle, \quad -3 \le g(x,y) \le 3,\ h(x,y) = 0$$

As $\nabla h(x,y) = \langle 0,1\rangle$ is never $\langle 0,0\rangle$, we get no points in this case.

Case B

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad -3 \le g(x,y) \le 3,\ h(x,y) = 0$$

We have $\langle 2x+y, 2y-6\rangle = \lambda\langle 0,1\rangle = \langle 0, \lambda\rangle$, so $2x+y=0$. As $y=0$, we have $x=0$. This is in the domain, so we get $\boxed{(0,0)}$ on the list.

(e) Lagrange critical points of the boundary piece #4,

$$\{-3\le x\le 3,\ y=5\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle, \quad -3 \le g(x,y) \le 3,\ h(x,y) = 5$$

As $\nabla h(x,y) = \langle 0,1\rangle$ is never $\langle 0,0\rangle$, we get no points in this case.

Case B

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad -3 \le g(x,y) \le 3,\ h(x,y) = 5$$

We have $\langle 2x+y, 2y-6\rangle = \lambda\langle 0,1\rangle = \langle 0,\lambda\rangle$, so $2x+y=0$. As $y=5$, $x=-\frac{5}{2}$.

As this is on the domain, we get $\boxed{(-\frac{5}{2},5)}$ on the list.

(f) All points of the boundary piece #5,

$$\{x=-3,\ y=0\}$$

This is just the point $\boxed{(-3,0)}$.

(g) All points of the boundary piece #6,

$$\{x=-3,\ y=5\}$$

This is just the point $\boxed{(-3,5)}$.

(h) All points of the boundary piece #7,

$$\{x=3,\ y=0\}$$

This is just the point $\boxed{(3,0)}$.

(i) All points of the boundary piece #8,

$$\{x=3,\ y=5\}$$

This is just the point $\boxed{(3,5)}$.

The list of values on the candidate points is:

- $f(-\frac{3}{2},3) = (-\frac{3}{2})^2 - \frac{3}{2}\cdot 3 + 3^2 - 18 = \frac{9}{4} + \frac{9}{2} - 18 = \frac{27}{4} - 18 = -\frac{45}{4}$
- $f(-3,3) = (-3)^2 - 3\cdot 3 + 3^2 - 18 = -9$
- $f(3,3) = 3^2 + 3\cdot 3 + 3^2 - 18 = 9$
- $f(0,0) = 0$
- $f(-\frac{5}{2},5) = (-\frac{5}{2})^2 - \frac{5}{2}\cdot 5 + 5^2 - 30 = \frac{25}{4} + \frac{25}{2} - 30 = \frac{75}{4} - 30 = -\frac{45}{4}$
- $f(-3,0) = 9$
- $f(-3,5) = (-3)^2 - 15 + 5^2 - 30 = -11$
- $f(3,0) = 9$
- $f(3,5) = 3^2 + 15 + 5^2 - 30 = 19$

So the global maximum value of f is 19, and the global minimum value of f is $-\frac{45}{4}$.

(5) There are 9 types of points we need to look for.

(a) Critcial points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x=0,\ 0 \le y \le 3\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{x=2,\ 0 \le y \le 3\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{0 \leq x \leq 2,\ y = 0\}$$

(e) Lagrange critical points of the boundary piece #4,

$$\{0 \leq x \leq 2,\ y = 3\}$$

(f) All points of the boundary piece #5,

$$\{x = 0,\ y = 0\}$$

(g) All points of the boundary piece #6,

$$\{x = 0,\ y = 3\}$$

(h) All points of the boundary piece #7,

$$\{x = 2,\ y = 0\}$$

(i) All points of the boundary piece #8,

$$\{x = 2,\ y = 3\}$$

Let $g(x, y) = x$, $h(x, y) = y$.

(a) Critcial points of the original domain.
The equations are

$$\nabla f(x, y) = \langle 0, 0\rangle, \quad 0 \leq g(x, y) \leq 2,\ 0 \leq h(x, y) \leq 3$$

As $\nabla f(x, y) = \langle 2x - 2, 4y - 4\rangle$, this being equal to $\langle 0, 0\rangle$ means $x = 1, y = 1$. As this is in the domain, we get $\boxed{(1, 1)}$ in the list.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = 0,\ 0 \leq y \leq 3\}$$

Case A The equations are

$$\nabla g(x, y) = \langle 0, 0\rangle, \quad g(x, y) = 0,\ 0 \leq h(x, y) \leq 3$$

As $\nabla g(x, y) = \langle 1, 0\rangle$ is never $\langle 0, 0\rangle$, there are no points from this case.

Case B

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0,\ 0 \leq h(x, y) \leq 3$$

We have $\langle 2x - 2, 4y - 4\rangle = \lambda\langle 1, 0\rangle = \langle \lambda, 0\rangle$, so $4y - 4 = 0$, or $y = 1$. As $x = 0$, and as this is on the domain, we get $\boxed{(0, 1)}$ on the list.

(c) Lagrange critical points of the boundary piece #2,

$$\{x = 2,\ 0 \leq y \leq 3\}$$

Case A The equations are

$$\nabla g(x, y) = \langle 0, 0\rangle, \quad g(x, y) = 2,\ 0 \leq h(x, y) \leq 3$$

As $\nabla g(x, y) = \langle 1, 0\rangle$ is never $\langle 0, 0\rangle$, there are no points from this case.

Case B

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 2,\ 0 \le h(x,y) \le 3$$

We have $\langle 2x-2, 4y-4\rangle = \lambda\langle 1,0\rangle = \langle \lambda, 0\rangle$, so $4y-4=0$, or $y=1$. As $x=2$, and as this is on the domain, we get $\boxed{(2,1)}$ on the list.

(d) Lagrange critical points of the boundary piece #3,

$$\{0 \le x \le 2,\ y = 0\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle, \quad 0 \le g(x,y) \le 2,\ h(x,y) = 0$$

As $\nabla h(x,y) = \langle 0,1\rangle$ is never $\langle 0,0\rangle$, there are no points from this case.

Case B

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad 0 \le g(x,y) \le 2,\ h(x,y) = 0$$

We have $\langle 2x-2, 4y-4\rangle = \lambda\langle 0,1\rangle = \langle 0, \lambda\rangle$, so $2x-2=0$, or $x=1$. As $y=0$, and as this is on the domain, we have $\boxed{(1,0)}$ on the domain.

(e) Lagrange critical points of the boundary piece #4,

$$\{0 \le x \le 2,\ y = 3\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle, \quad 0 \le g(x,y) \le 2,\ h(x,y) = 3$$

As $\nabla h(x,y) = \langle 0,1\rangle$ is never $\langle 0,0\rangle$, there are no points from this case.

Case B

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad 0 \le g(x,y) \le 2,\ h(x,y) = 3$$

We have $\langle 2x-2, 4y-4\rangle = \lambda\langle 0,1\rangle = \langle 0, \lambda\rangle$, so $2x-2=0$, or $x=1$. As $y=3$, and as this is on the domain, we have $\boxed{(1,3)}$ on the list.

(f) All points of the boundary piece #5,

$$\{x = 0,\ y = 0\}$$

This is just the point $\boxed{(0,0)}$.

(g) All points of the boundary piece #6,

$$\{x = 0,\ y = 3\}$$

This is just the point $\boxed{(0,3)}$.

(h) All points of the boundary piece #7,

$$\{x = 2,\ y = 0\}$$

This is just the point $\boxed{(2,0)}$.

(i) All points of the boundary piece #8,

$$\{x = 2,\ y = 3\}$$

This is just the point $\boxed{(2,3)}$.

The list of values on the candidate points is:

- $f(1,1) = 1+2-2-4+1 = -2$

- $f(0,1) = 0 + 2 - 0 - 4 + 1 = -1$
- $f(2,1) = 4 + 2 - 4 - 4 + 1 = -1$
- $f(1,0) = 1 + 0 - 2 - 0 + 1 = 0$
- $f(1,3) = 1 + 18 - 2 - 12 + 1 = 6$
- $f(0,0) = 0 + 0 - 0 - 0 + 1 = 1$
- $f(0,3) = 0 + 18 - 0 - 12 + 1 = 7$
- $f(2,0) = 4 + 0 - 4 - 0 + 1 = 1$
- $f(2,3) = 4 + 18 - 4 - 12 + 1 = 7$

So the global maximum value of $f(x,y)$ on D is 7, and the global minimum value of $f(x,y)$ on D is -2.

□

Exercise 3. Find the global maximum and minimum values of f on the given domain.

(1) $f(x,y) = x^2y$, on the domain $\{(x,y) \mid x^2+y^2=1,\ y \geq 0\}$.
(2) $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$, on the domain $\{(x,y) \mid x^2+y^2=4,\ x+y \geq 0\}$.
(3) $f(x,y,z) = xyz$, on the domain $\{(x,y,z) \mid x^2+y^2+z^2=3,\ z \geq 0\}$.

Solution.

(1) There are two types of points we need to look for.
 (a) Lagrange critical points of the original domain.
 (b) All points of the boundary piece #1,

$$\{x^2+y^2=1,\ y=0\}$$

Let $g(x,y) = x^2+y^2$, $h(x,y) = y$.
 (a) Lagrange critical points of the original domain.
 Case A The equations are

$$\nabla g(x,y) = \langle 0,0 \rangle, \quad g(x,y) = 1,\ h(x,y) \geq 0$$

Since $\nabla g(x,y) = \langle 2x, 2y \rangle$, this is $\langle 0,0 \rangle$ precisely when $x=y=0$, which does not satisfy $x^2+y^2=1$. Thus there are no points in this case.
 Case B The equations are

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 1,\ h(x,y) \geq 0$$

As $\nabla f(x,y) = \langle 2xy, x^2 \rangle$, we have $\langle 2xy, x^2 \rangle = \lambda \langle 2x, 2y \rangle = \langle 2\lambda x, 2\lambda y \rangle$. Thus we get the system of equations

$$\boxed{Eq1} \cdots 2xy = 2\lambda x$$

$$\boxed{Eq2} \cdots x^2 = 2\lambda y$$

$$\boxed{Eq3} \cdots x^2+y^2 = 1$$

$$\boxed{Eq4} \cdots y \geq 0$$

From $\boxed{Eq1}$, we want to divide by $2x$, and get $\boxed{y=\lambda}$, but this may not be possible if $\boxed{x=0}$. So we get two cases.

- The case of $\boxed{y=\lambda}$. We could then plug it into $\boxed{Eq2}$ and get $x^2 = 2y^2$. As $x^2+y^2=1$, we have $3y^2=1$, or $y^2=\frac{1}{3}$. Thus either $y=\frac{1}{\sqrt{3}}$ or $y=-\frac{1}{\sqrt{3}}$. As we require $y \geq 0$, only $y=\frac{1}{\sqrt{3}}$ is possible. Also, as $x^2=2y^2=\frac{2}{3}$, either $x=\frac{\sqrt{2}}{\sqrt{3}}$ or $x=-\frac{\sqrt{2}}{\sqrt{3}}$. Thus we get the points $\boxed{(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})}$ and $\boxed{(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})}$ on the list.
- The case of $\boxed{x=0}$. This you may then plug into $x^2+y^2=1$, and get $y^2=1$. Thus either $y=1$ or $y=-1$. As $y \geq 0$ is a requirement, only $\boxed{(0,1)}$ is on the list from this acse.

(b) All points of the boundary piece #1,

$$\{x^2+y^2=1,\ y=0\}$$

As $y=0$, this means $x^2=1$, so either $x=1$ or $x=-1$. Thus we get $\boxed{(1,0)}$, $\boxed{(-1,0)}$ on the list.

The list of values on the candidate points:

- $f(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$
- $f(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$
- $f(0,1)=0$
- $f(1,0)=0$
- $f(-1,0)=0$

So the global maximum value is $\frac{2}{3\sqrt{3}}$ and the global minimum value is 0.

(2) There are two types of points we need to look for.

(a) Lagrange critical points of the original domain.

(b) All points of the boundary piece #1,

$$\{x^2+y^2=4,\ x+y=0\}$$

Let $g(x,y)=x^2+y^2$, $h(x,y)=x+y$.

(a) Lagrange critical points of the original domain.

Case A The equations are

$$\nabla g(x,y)=\langle 0,0\rangle, \quad g(x,y)=4,\ h(x,y)\geq 0$$

As $\nabla g(x,y)=\langle 2x,2y\rangle$, this is equal to $\langle 0,0\rangle$ precisely when $x=y=0$, which does not satisfy $x^2+y^2=4$. Thus, there is no point from this case.

Case B The equations are

$$\nabla f(x,y)=\lambda\nabla g(x,y), \quad g(x,y)=4,\ h(x,y)\geq 0$$

As

$$\begin{aligned}\nabla f(x,y) &= \langle -2xe^{-x^2-y^2}(x^2+2y^2)+2xe^{-x^2-y^2}, -2ye^{-x^2-y^2}(x^2+2y^2)+4ye^{-x^2-y^2}\rangle \\ &= \langle -2xe^{-x^2-y^2}(x^2+2y^2-1), -2ye^{-x^2-y^2}(x^2+2y^2-2)\rangle\end{aligned}$$

We have the following system of equations.

$$\boxed{Eq1}\cdots -2xe^{-x^2-y^2}(x^2+2y^2-1)=2\lambda x$$

$$\boxed{Eq2}\cdots -2ye^{-x^2-y^2}(x^2+2y^2-2)=2\lambda y$$

$$\boxed{Eq3} \cdots x^2 + y^2 = 4$$

$$\boxed{Eq4} \cdots x + y \geq 0$$

From $\boxed{Eq1}$, you may want to divide by $2x$ and get $\boxed{\lambda = -e^{-x^2-y^2}(x^2 + 2y^2 - 1)}$. This may not be possible if $\boxed{x = 0}$.

- The case of $\boxed{\lambda = -e^{-x^2-y^2}(x^2 + 2y^2 - 1)}$. You then plug into $\boxed{Eq2}$ and get

  $-2ye^{-x^2-y^2}(x^2 + 2y^2 - 2) = -2ye^{-x^2-y^2}(x^2 + 2y^2 - 1)$

  This implies that $2ye^{-x^2-y^2} = 0$. Since $e^{-x^2-y^2} \neq 0$, we get $y = 0$. Plugging into $\boxed{Eq3}$, we get $x^2 = 4$, which means either $x = 2$ or $x = -2$. As we need $x + y \geq 0$, the only possibility is $\boxed{(2, 0)}$.
- The case of $\boxed{x = 0}$. You plug into $\boxed{Eq3}$ and get $y^2 = 4$, which means either $y = 2$ or $y = -2$. As we need $x + y \geq 0$, the only possibility is $\boxed{(0, 2)}$.

(b) All points of the boundary piece #1,

$$\{x^2 + y^2 = 4, \ x + y = 0\}$$

As $x + y = 0$, we have $x = -y$, so $2x^2 = 4$, or $x^2 = 2$. Thus, either $x = \sqrt{2}$ (in which case $y = -\sqrt{2}$) or $x = -\sqrt{2}$ (in which case $y = \sqrt{2}$). Thus we get $\boxed{(\sqrt{2}, -\sqrt{2})}$ and $\boxed{(-\sqrt{2}, \sqrt{2})}$ on the list.

The list of values on the candidate points:

- $f(2, 0) = 4e^{-4}$
- $f(0, 2) = 8e^{-4}$
- $f(\sqrt{2}, -\sqrt{2}) = 6e^{-4}$
- $f(-\sqrt{2}, \sqrt{2}) = 6e^{-4}$

So, the global maximum value is $8e^{-4}$ and the global minimum value is $4e^{-4}$.

(3) There are two types of points we need to look for.

(a) Lagrange critical points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x^2 + y^2 + z^2 = 3, \ z = 0\}$$

Let $g(x, y, z) = x^2 + y^2 + z^2$, $h(x, y, z) = z$.

(a) Lagrange critical points of the original domain.

Case A The equations are

$$\nabla g(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) = 3, \ h(x, y, z) \geq 0$$

As $\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$, this is $\langle 0, 0, 0 \rangle$ precisely when $x = y = z = 0$, which does not satisfy $x^2 + y^2 + z^2 = 3$. Thus, there are no points from this case.

Case B The equations are

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \quad g(x, y, z) = 3, \ h(x, y, z) \geq 0$$

As $\nabla f(x,y,z) = \langle yz, xz, xy \rangle$, we get the system of equations

$$\boxed{Eq1} \cdots yz = 2\lambda x$$

$$\boxed{Eq2} \cdots xz = 2\lambda y$$

$$\boxed{Eq3} \cdots xy = 2\lambda z$$

$$\boxed{Eq4} \cdots x^2 + y^2 + z^2 = 3$$

$$\boxed{Eq5} \cdots z \geq 0$$

By taking x times $\boxed{Eq1}$, y times $\boxed{Eq2}$, and z times $\boxed{Eq3}$, we get

$$xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2.$$

You want to divide by 2λ and get $\boxed{x^2 = y^2 = z^2}$, but this may not be possible if $\boxed{\lambda = 0}$.

- The case of $\boxed{x^2 = y^2 = z^2}$. Then, $x^2 + y^2 + z^2 = 3$ becomes $3x^2 = 3$, or $x^2 = y^2 = z^2 = 1$. So, each x, y, z is either 1 or -1. As $z \geq 0$, z must be 1. Therefore, there are four points on the list from this case, $\boxed{(1,1,1)}$, $\boxed{(1,-1,1)}$, $\boxed{(-1,1,1)}$, $\boxed{(-1,-1,1)}$.
- The case of $\boxed{\lambda = 0}$. This means that $yz = xy = xz = 0$. This means that at least two of the three numbers x, y, z are zero. Thus, either $x = y = 0$ or $x = z = 0$ or $y = z = 0$. Each case, using $x^2 + y^2 + z^2 = 3$ and $z \geq 0$, we get the points $\boxed{(0,0,\sqrt{3})}$, $\boxed{(0,\sqrt{3},0)}$, $\boxed{(0,-\sqrt{3},0)}$, $\boxed{(\sqrt{3},0,0)}$, $\boxed{(-\sqrt{3},0,0)}$.

(b) Lagrange critical points of the boundary piece #1,

$$\{x^2 + y^2 + z^2 = 3,\ z = 0\}$$

Case A1 The equations are

$$\nabla g(x,y,z) = \langle 0,0,0 \rangle, \quad g(x,y,z) = 3,\ h(x,y,z) = 0$$

As $\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$, this is $\langle 0,0,0 \rangle$ precisely when $x = y = z = 0$, which does not satisfy $x^2 + y^2 + z^2 = 3$. Thus, there are no points from this case.

Case A2 The equations are

$$\nabla h(x,y,z) = \langle 0,0,0 \rangle, \quad g(x,y,z) = 3,\ h(x,y,z) = 0$$

As $\nabla h(x,y,z) = \langle 0,0,1 \rangle \neq \langle 0,0,0 \rangle$, there are no points from this case.

Case B The equations are

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), \quad g(x,y,z) = 3,\ h(x,y,z) = 0$$

We get the system of equations

$$\boxed{Eq1} \cdots yz = 2\lambda x$$

$$\boxed{Eq2} \cdots xz = 2\lambda y$$

$$\boxed{Eq3} \cdots xy = 2\lambda z + \mu$$
$$\boxed{Eq4} \cdots x^2 + y^2 + z^2 = 3$$
$$\boxed{Eq5} \cdots z = 0$$

As $z = 0$, these become simpler:

$$\boxed{Eq1} \cdots 0 = 2\lambda x$$
$$\boxed{Eq2} \cdots 0 = 2\lambda y$$
$$\boxed{Eq3} \cdots xy = \mu$$
$$\boxed{Eq4} \cdots x^2 + y^2 = 3$$

In fact, $\lambda = 0$ and $\mu = xy$ makes the equations satisfied as long as $x^2 + y^2 = 3$. Thus, we see that all points $\boxed{(x, y, 0)}$ with $x^2 + y^2 = 3$ are Lagrange critical points.

The list of values on the candidate points:

- $f(1, 1, 1) = 1$
- $f(1, -1, 1) = -1$
- $f(-1, 1, 1) = -1$
- $f(-1, -1, 1) = 1$
- $f(0, 0, \sqrt{3}) = 0$
- $f(0, \sqrt{3}, 0) = 0$
- $f(0, -\sqrt{3}, 0) = 0$
- $f(\sqrt{3}, 0, 0) = 0$
- $f(-\sqrt{3}, 0, 0) = 0$
- For $x^2 + y^2 = 3$, $f(x, y, 0) = 0$.

The global max value is 1 and the global min value is -1.

$\square$

Exercise 4. Find the global maximum and minimum values of f on the given domain.

(1) $f(x, y) = x^3 - 12x + y^3 - 12y$ on the domain
$$D = \{(x, y) \mid (x + 2)^2 + (y + 2)^2 \leq 13,\ x \geq -5\}$$

(2) $f(x, y) = x + y$ on the domain
$$D = \{(x, y) \mid 0 \leq x \leq 1,\ ex^2 \leq y \leq e^x\}$$

(3) $f(x, y, z) = x^4 + y + z^2$ on the domain
$$D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq \frac{1}{4},\ x \geq 0\}$$

(4) $f(x, y, z) = xz + yz - xy$ on the domain
$$D = \{(x, y, z) \mid z^2 \geq x^2 + y^2,\ z^2 \leq 4\}$$

Solution.

(1) There are four types of points we need to look for.
  (a) Critical points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{(x+2)^2+(y+2)^2=13,\ x\geq -5\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{(x+2)^2+(y+2)^2\leq 13,\ x=-5\}$$

(d) All points of the boundary piece #3,

$$\{(x+2)^2+(y+2)^2=13,\ x=-5\}$$

Let $g(x,y)=(x+2)^2+(y+2)^2$, $h(x,y)=x$.

(a) Critical points of the original domain.
The equations are

$$\nabla f(x,y)=\langle 0,0\rangle,\quad g(x,y)\leq 13,\ h(x,y)\geq -5$$

As $\nabla f(x,y)=\langle 3x^2-12,3y^2-12\rangle$, it is equal to $\langle 0,0\rangle$ precisely when $3x^2-12=0$ and $3y^2-12=0$. This means $x^2=4$ and $y^2=4$, so both x and y are either 2 or -2. All these points satisfy $x\geq -5$, so we get four points on the list, $\boxed{(-2,-2)}$, $\boxed{(-2,2)}$, $\boxed{(2,-2)}$, $\boxed{(2,2)}$.

(b) Lagrange critical points of the boundary piece #1,

$$\{(x+2)^2+(y+2)^2=13,\ x\geq -5\}$$

Case A The equations are

$$\nabla g(x,y)=\langle 0,0\rangle,\quad g(x,y)=13,\ h(x,y)\geq -5$$

As $\nabla g(x,y)=\langle 2(x+2),2(y+2)\rangle$, this is equal to $\langle 0,0\rangle$ precisely when $x+2=y+2=0$. This does not satisfy $(x+2)^2+(y+2)^2=13$, so there are no points in this case.

Case B The equations are

$$\nabla f(x,y)=\lambda\nabla g(x,y),\quad g(x,y)=13,\ h(x,y)\geq -5$$

We have the system of equations

$$\boxed{Eq1}\cdots 3x^2-12=2\lambda(x+2)$$

$$\boxed{Eq2}\cdots 3y^2-12=2\lambda(y+2)$$

$$\boxed{Eq3}\cdots (x+2)^2+(y+2)^2=13$$

$$\boxed{Eq4}\cdots x\geq -5$$

Note that $\boxed{Eq1}$ is $3(x-2)(x+2)=3(x^2-4)=2\lambda(x+2)$, so we want to divide by $2(x+2)$ on both sides and get $\boxed{\lambda=\frac{3}{2}(x-2)}$. This may not be possible if $\boxed{x=-2}$.

- The case of $\boxed{\lambda = \frac{3}{2}(x-2)}$. Plugging into $\boxed{Eq2}$, we get

  $$3(y-2)(y+2) = 3y^2 - 12 = 3(x-2)(y+2)$$

  Thus, $3(x-y)(y+2) = 0$. Therefore, either $\boxed{x = y}$ or $\boxed{y = -2}$.

  * The case of $\boxed{x = y}$. Plugging into $\boxed{Eq3}$, we get $2(x+2)^2 = 13$, so $x + 2 = \pm\frac{\sqrt{13}}{\sqrt{2}}$, so x is either $-2 + \frac{\sqrt{13}}{\sqrt{2}}$ or $-2 - \frac{\sqrt{13}}{\sqrt{2}}$. Both cases satisfy $x \geq -5$, so we get two points on the list, $\boxed{(-2 + \frac{\sqrt{13}}{\sqrt{2}}, -2 + \frac{\sqrt{13}}{\sqrt{2}})}$, $\boxed{(-2 - \frac{\sqrt{13}}{\sqrt{2}}, -2 - \frac{\sqrt{13}}{\sqrt{2}})}$.
  * The case of $\boxed{y = -2}$. Plugging into $\boxed{Eq3}$, we get $(x+2)^2 = 13$, so $x + 2 = \pm\sqrt{13}$, so either $x = -2 + \sqrt{13}$ or $x = -2 - \sqrt{13}$. Note that $-2 + \sqrt{13} \geq -5$ while $-2 - \sqrt{13} < -5$, so we only get one point, $\boxed{(-2 + \sqrt{13}, -2)}$.

- The case of $\boxed{x = -2}$. Plugging into $\boxed{Eq3}$, we get $(y+2)^2 = 13$, so either $y = -2 + \sqrt{13}$ or $y = -2 - \sqrt{13}$. Thus we get two points, $\boxed{(-2, -2 + \sqrt{13})}$, $\boxed{(-2, -2 - \sqrt{13})}$.

(c) Lagrange critical points of the boundary piece #2,

$$\{(x+2)^2 + (y+2)^2 \leq 13,\ x = -5\}$$

Case A The equations are

$$\nabla h(x, y) = \langle 0, 0\rangle, \quad g(x, y) \leq 13,\ h(x, y) = -5$$

As $\nabla h(x, y) = \langle 1, 0\rangle \neq \langle 0, 0\rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x, y) = \lambda \nabla h(x, y), \quad g(x, y) \leq 13,\ h(x, y) = -5$$

This means $\langle 3x^2 - 12, 3y^2 - 12\rangle = \lambda\langle 1, 0\rangle = \langle \lambda, 0\rangle$, so $3y^2 - 12 = 0$, or $y^2 = 4$. This means $y = 2$ or $y = -2$. Note that $(x, y) = (-5, -2)$ satisfies $(x+2)^2 + (y+2)^2 \leq 13$ while $(x, y) = (-5, 2)$ does not, so we get one point $\boxed{(-5, -2)}$ on the list.

(d) All points of the boundary piece #3,

$$\{(x+2)^2 + (y+2)^2 = 13,\ x = -5\}$$

As $x = -5$, we get $(y+2)^2 = 13 - 9 = 4$, so $y + 2$ is either 2 or -2, which means $y = 0$ or $y = -4$. So we get the points $\boxed{(-5, 0)}$ and $\boxed{(-5, -4)}$ on the list.

The list of values on the candidate points:

- $f(-2, -2) = 32$
- $f(-2, 2) = 0$
- $f(2, -2) = 0$

- $f(2,2) = -32$
- $f(-2+\frac{\sqrt{13}}{\sqrt{2}}, -2+\frac{\sqrt{13}}{\sqrt{2}}) = -46+13\sqrt{13}$
- $f(-2-\frac{\sqrt{13}}{\sqrt{2}}, -2-\frac{\sqrt{13}}{\sqrt{2}}) = -46-13\sqrt{13}$
- $f(-2+\sqrt{13}, -2) = -46+13\sqrt{13}$
- $f(-2, -2+\sqrt{13}) = -46+13\sqrt{13}$
- $f(-2, -2-\sqrt{13}) = -46-13\sqrt{13}$
- $f(-5,-2) = -49$
- $f(-5,0) = -65$
- $f(-5,-4) = -81$

The global max value is 32 and the global min value is $-46-13\sqrt{13}$.

(2) There are 10 types of points we need to look for.
(a) Critical points of the original domain.
(b) Lagrange critical points of the boundary piece #1,
$$\{x=0,\ ex^2 \le y \le e^x\}$$
(c) Lagrange critical points of the boundary piece #2,
$$\{x=1,\ ex^2 \le y \le e^x\}$$
(d) Lagrange critical points of the boundary piece #3,
$$\{0 \le x \le 1,\ y = ex^2\}$$
(e) Lagrange critical points of the boundary piece #4,
$$\{0 \le x \le 1,\ y = e^x\}$$
(f) All points of the boundary piece #5,
$$\{x=0,\ y=ex^2\}$$
(g) All points of the boundary piece #6,
$$\{x=0,\ y=e^x\}$$
(h) All points of the boundary piece #7,
$$\{x=1,\ y=ex^2\}$$
(i) All points of the boundary piece #8,
$$\{x=1,\ y=e^x\}$$
(j) All points of the boundary piece #9,
$$\{0 \le x \le 1,\ y = ex^2 = e^x\}$$
(Here we get this boundary piece considered, because it is not clear at first sight whether $ex^2 = e^x$ is a bogus empty condition or not)

Let $g(x,y) = x$, $h(x,y) = y - ex^2$, $i(x,y) = y - e^x$, so that the original domain is expressed using $g(x,y) \ge 0$, $g(x,y) \le 1$, $h(x,y) \ge 0$, $i(x,y) \le 0$.

(a) Critical points of the original domain.
The equations are

$$\nabla f(x,y) = \langle 0,0\rangle, \quad 0 \le x \le 1,\ ex^2 \le y \le e^x$$

As $\nabla f(x,y) = \langle 1,1\rangle \neq \langle 0,0\rangle$, there are no points in this case.

(b) Lagrange critical points of the boundary piece #1,

$$\{x = 0,\ ex^2 \le y \le e^x\}$$

Case A The equations are

$$\nabla g(x,y) = \langle 0,0\rangle, \quad x = 0,\ ex^2 \le y \le e^x$$

As $\nabla g(x,y) = \langle 1,0\rangle \neq \langle 0,0\rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad x = 0,\ ex^2 \le y \le e^x$$

We have $\langle 1,1\rangle = \lambda\langle 1,0\rangle = \langle \lambda,0\rangle$, which is impossible, so there are no points in this case.

(c) Lagrange critical points of the boundary piece #2,

$$\{x = 1,\ ex^2 \le y \le e^x\}$$

Case A The equations are

$$\nabla g(x,y) = \langle 0,0\rangle, \quad x = 1,\ ex^2 \le y \le e^x$$

As $\nabla g(x,y) = \langle 1,0\rangle \neq \langle 0,0\rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad x = 1,\ ex^2 \le y \le e^x$$

We have $\langle 1,1\rangle = \lambda\langle 1,0\rangle = \langle \lambda,0\rangle$, which is impossible, so there are no points in this case.

(d) Lagrange critical points of the boundary piece #3,

$$\{0 \le x \le 1,\ y = ex^2\}$$

Case A The equations are

$$\nabla h(x,y) = \langle 0,0\rangle, \quad 0 \le x \le 1,\ y = ex^2$$

As $\nabla h(x,y) = \langle -2ex,1\rangle \neq \langle 0,0\rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla h(x,y), \quad 0 \le x \le 1,\ y = ex^2$$

We have

$$\langle 1,1\rangle = \lambda\langle -2ex,1\rangle = \langle -2e\lambda x,\lambda\rangle$$

Thus, $\lambda = 1$ and $-2ex = 1$, which means $x = -\frac{1}{2e}$. As we want $0 \le x \le 1$, this is impossible, yielding no points in this case.

(e) Lagrange critical points of the boundary piece #4,

$$\{0 \le x \le 1,\ y = e^x\}$$

Case A The equations are

$$\nabla i(x,y) = \langle 0,0\rangle, \quad 0 \le x \le 1,\ y = e^x$$

As $\nabla i(x,y) = \langle -e^x, 1\rangle \ne \langle 0,0\rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x,y) = \lambda \nabla i(x,y), \quad 0 \le x \le 1,\ y = e^x$$

We have $\langle 1,1\rangle = \lambda\langle -e^x, 1\rangle = \langle -\lambda e^x, \lambda\rangle$. Thus, $\lambda = 1$ and $-e^x = 1$, which is impossible as e^x is always positive. So, there are no points in this case.

(f) All points of the boundary piece #5,

$$\{x = 0,\ y = ex^2\}$$

This is the point $\boxed{(0,0)}$.

(g) All points of the boundary piece #6,

$$\{x = 0,\ y = e^x\}$$

This is the point $\boxed{(0,1)}$.

(h) All points of the boundary piece #7,

$$\{x = 1,\ y = ex^2\}$$

This is the point $\boxed{(1,e)}$

(i) All points of the boundary piece #8,

$$\{x = 1,\ y = e^x\}$$

This is the point $\boxed{(1,e)}$.

(j) All points of the boundary piece #9,

$$\{0 \le x \le 1,\ y = ex^2 = e^x\}$$

This means $ex^2 = e^x$, but $ex^2 \le e^x$ for all $0 \le x \le 1$, and this meets when $x = 1$. So this means $x = 1$ and $y = e$, so again the point $\boxed{(1,e)}$.

The list of values on the candidate points:

- $f(0,0) = 0$
- $f(0,1) = 1$
- $f(1,e) = 1 + e$

The global max value is $e + 1$ and the global min value is 0.

(3) There are four types of points we need to look for.

(a) Critical points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{x^2 + y^2 + z^2 = \frac{1}{4},\ x \ge 0\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{x^2 + y^2 + z^2 \le \frac{1}{4},\ x = 0\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{x^2 + y^2 + z^2 = \frac{1}{4},\ x = 0\}$$

Let $g(x, y, z) = x^2 + y^2 + z^2$, $h(x, y, z) = x$.

(a) Critical points of the original domain.
The equations are

$$\nabla f(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) \le \frac{1}{4},\ h(x, y, z) \ge 0$$

As $\nabla f(x, y, z) = \langle 4x^3, 1, 2z \rangle \neq \langle 0, 0, 0 \rangle$, there are no points in this case.

(b) Lagrange critical points of the boundary piece #1,

$$\{x^2 + y^2 + z^2 = \frac{1}{4},\ x \ge 0\}$$

Case A The equations are

$$\nabla g(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) = \frac{1}{4},\ h(x, y, z) \ge 0$$

As $\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$, this is $\langle 0, 0, 0 \rangle$ precisely when $x = y = z = 0$, which does not satisfy $x^2 + y^2 + z^2 = \frac{1}{4}$. Thus there are no points in this case.

Case B The equations are

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \quad g(x, y, z) = \frac{1}{4},\ h(x, y, z) \ge 0$$

We get the system of equations

$$\boxed{Eq1} \cdots 4x^3 = 2\lambda x$$

$$\boxed{Eq2} \cdots 1 = 2\lambda y$$

$$\boxed{Eq3} \cdots 2z = 2\lambda z$$

$$\boxed{Eq4} \cdots x^2 + y^2 + z^2 = \frac{1}{4}$$

$$\boxed{Eq5} \cdots x \ge 0$$

We want to divide $\boxed{Eq3}$ by $2z$ and get $\boxed{\lambda = 0}$. This may not be possible if $\boxed{z = 0}$.

- The case of $\boxed{\lambda = 0}$. This is impossible because $\boxed{Eq2}$ becomes $1 = 0$. So there are no points in this case.
- The case of $\boxed{z = 0}$. Then the equations become

$$\boxed{Eq1} \cdots 4x^3 = 2\lambda x$$

$$\boxed{Eq2} \cdots 1 = 2\lambda y$$

$$\boxed{Eq4} \cdots x^2 + y^2 = \frac{1}{4}$$

$$\boxed{Eq5} \cdots x \ge 0$$

We want to divide $\boxed{Eq1}$ by $2x$ and get $\boxed{\lambda = 2x^2}$. This may not be possible if $\boxed{x = 0}$.

- ∗ The case of $\boxed{\lambda = 2x^2}$. In this case $\boxed{Eq2}$ becomes $1 = 4x^2y$, so $y > 0$. In particular, we may divide by $4y$ and get $x^2 = \frac{1}{4y}$. As $x^2 + y^2 = \frac{1}{4}$, we get $\frac{1}{4y} + y^2 = \frac{1}{4}$.
  How can this ever happen? If $y \geq 1$, then $\frac{1}{4y} + y^2 \geq y^2 \geq 1$, so this cannot be equal to $\frac{1}{4}$. On the other hand, if $0 < y \leq 1$, then $\frac{1}{4y} + y^2 > \frac{1}{4y} \geq \frac{1}{4}$, so again this cannot be equal to $\frac{1}{4}$. So, this equality can never be satisfied, and there are no points in this case.
- ∗ The case of $\boxed{x = 0}$. This means $x = z = 0$, so $y^2 = \frac{1}{4}$, so either $y = \frac{1}{2}$ or $y = -\frac{1}{2}$. So we have two points on the list, $\boxed{(0, \frac{1}{2}, 0)}$ and $\boxed{(0, -\frac{1}{2}, 0)}$.

(c) Lagrange critical points of the boundary piece #2,

$$\{x^2 + y^2 + z^2 \leq \frac{1}{4},\ x = 0\}$$

Case A The equations are

$$\nabla h(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) \leq \frac{1}{4},\ h(x, y, z) = 0$$

As $\nabla h(x, y, z) = \langle 1, 0, 0 \rangle \neq \langle 0, 0, 0 \rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x, y, z) = \lambda \nabla h(x, y, z), \quad g(x, y, z) \leq \frac{1}{4},\ h(x, y, z) = 0$$

We have $\langle 4x^3, 1, 2z \rangle = \lambda \langle 1, 0, 0 \rangle = \langle \lambda, 0, 0 \rangle$, which is impossible because of the second component. So there are no points in this case.

(d) Lagrange critical points of the boundary piece #3,

$$\{x^2 + y^2 + z^2 = \frac{1}{4},\ x = 0\}$$

Case A1 The equations are

$$\nabla g(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) = \frac{1}{4},\ h(x, y, z) = 0$$

As $\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$, this is $\langle 0, 0, 0 \rangle$ precisely when $x = y = z = 0$, which does not satisfy $x^2 + y^2 + z^2 = \frac{1}{4}$. Thus there are no points in this case.

Case A2 The equations are

$$\nabla h(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) = \frac{1}{4},\ h(x, y, z) = 0$$

As $\nabla h(x, y, z) = \langle 1, 0, 0 \rangle \neq \langle 0, 0, 0 \rangle$, there are no points in this case.

Case B The equations are

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z), \quad g(x, y, z) = \frac{1}{4},\ h(x, y, z) = 0$$

We get the system of equations

$$\boxed{Eq1} \cdots 4x^3 = 2\lambda x + \mu$$

$$\boxed{Eq2} \cdots 1 = 2\lambda y$$

$$\boxed{Eq3} \cdots 2z = 2\lambda z$$

$$\boxed{Eq4} \cdots x^2 + y^2 + z^2 = \frac{1}{4}$$

$$\boxed{Eq5} \cdots x = 0$$

As $x = 0$, these simplify into

$$\boxed{Eq1} \cdots 0 = \mu$$

$$\boxed{Eq2} \cdots 1 = 2\lambda y$$

$$\boxed{Eq3} \cdots 2z = 2\lambda z$$

$$\boxed{Eq4} \cdots y^2 + z^2 = \frac{1}{4}$$

We want to divide $\boxed{Eq3}$ by $2z$ and get $\boxed{\lambda = 1}$, which may not be possible if $\boxed{z = 0}$.

- The case of $\boxed{\lambda = 1}$. In this case $\boxed{Eq2}$ says $2y = 1$, or $y = \frac{1}{2}$. Then by $\boxed{Eq4}$, $z^2 = 0$, so $z = 0$. So again we get the point $\boxed{(0, \frac{1}{2}, 0)}$.
- The case of $\boxed{z = 0}$. Then $\boxed{Eq4}$ becomes $y^2 = \frac{1}{4}$, so we get two points on the list, $\boxed{(0, \frac{1}{2}, 0)}$, $\boxed{(0, -\frac{1}{2}, 0)}$.

The list of values on the candidate points:

- $f(0, \frac{1}{2}, 0) = \frac{1}{2}$
- $f(0, -\frac{1}{2}, 0) = -\frac{1}{2}$

So the global maximum value is $\frac{1}{2}$ and the global minimum value is $-\frac{1}{2}$.

(4) There are four types of points we need to look for.

(a) Critical points of the original domain.

(b) Lagrange critical points of the boundary piece #1,

$$\{z^2 = x^2 + y^2,\ z^2 \leq 4\}$$

(c) Lagrange critical points of the boundary piece #2,

$$\{z^2 \geq x^2 + y^2,\ z^2 = 4\}$$

(d) Lagrange critical points of the boundary piece #3,

$$\{z^2 = x^2 + y^2,\ z^2 = 4\}$$

Let $g(x, y, z) = z^2 - x^2 - y^2$ and $h(x, y, z) = z^2$.

(a) Critical points of the original domain.
The equations are

$$\nabla f(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) \geq 0, \ h(x, y, z) \leq 4$$

As $\nabla f(x, y, z) = \langle z - y, z - x, x + y \rangle$, this is $\langle 0, 0, 0 \rangle$ if $z - y = 0$, $z - x = 0$ and $x + y = 0$. This means $x = y = z$ and $x + y = 0$. So, $x = y = z = 0$. This point satisfies the domain inequalities, so we get a point on the list $\boxed{(0, 0, 0)}$.

(b) Lagrange critical points of the boundary piece #1,

$$\{z^2 = x^2 + y^2, \ z^2 \leq 4\}$$

Case A The equations are

$$\nabla g(x, y, z) = \langle 0, 0, 0 \rangle, \quad g(x, y, z) = 0, \ h(x, y, z) \leq 4$$

As $\nabla g(x, y, z) = \langle -2x, -2y, 2z \rangle$, it is $\langle 0, 0, 0 \rangle$ if $x = y = z = 0$. This satisfies the domain equations, so $\boxed{(0, 0, 0)}$ is a point on the list.

Case B The equations are

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \quad g(x, y, z) = 0, \ h(x, y, z) \leq 4$$

We get the system of equations

$$\boxed{Eq1} \cdots z - y = -2\lambda x$$

$$\boxed{Eq2} \cdots z - x = -2\lambda y$$

$$\boxed{Eq3} \cdots x + y = 2\lambda z$$

$$\boxed{Eq4} \cdots z^2 = x^2 + y^2$$

$$\boxed{Eq5} \cdots z^2 \leq 4$$

We compute $\boxed{Eq1} - \boxed{Eq2}$ and get

$$x - y = -2\lambda(x - y)$$

We want to divide this by $x - y$ and get $\boxed{\lambda = -\frac{1}{2}}$, which may not be possible if $\boxed{x = y}$.

- The case of $\boxed{\lambda = -\frac{1}{2}}$. By plugging into the equations we get

$$\boxed{Eq1} \cdots z - y = x$$

$$\boxed{Eq2} \cdots z - x = y$$

$$\boxed{Eq3} \cdots x + y = -z$$

$$\boxed{Eq4} \cdots z^2 = x^2 + y^2$$

$$\boxed{Eq5} \cdots z^2 \leq 4$$

So on one hand $\boxed{Eq1}$ says $z = x + y$ but on the other hand $\boxed{Eq3}$ says $z = -x - y$. So $z = x + y = 0$. This together with $z^2 = x^2 + y^2$ implies that $x = y = z = 0$. So we get $\boxed{(0,0,0)}$ on the list from this case.

– The case of $\boxed{x = y}$. Plugging into the equations we get

$$\boxed{Eq1} \cdots z - x = -2\lambda x$$

$$\boxed{Eq3} \cdots 2x = 2\lambda z$$

$$\boxed{Eq4} \cdots z^2 = 2x^2$$

$$\boxed{Eq5} \cdots z^2 \leq 4$$

So $\boxed{Eq1}$ says $z = (1-2\lambda)x$ and $\boxed{Eq3}$ says $x = \lambda z$. Thus $z = (1-2\lambda)x = \lambda(1-2\lambda)z$. This is possible either when $\boxed{z = 0}$ or $\boxed{\lambda(1-2\lambda) = 1}$.

* The case of $\boxed{z = 0}$. This implies that $x^2 = 0$, so $x = y = z = 0$, so we again get the point $\boxed{(0,0,0)}$.
* The case of $\boxed{\lambda(1-2\lambda) = 1}$. This is $-2\lambda^2 + \lambda = 1$, or $2\lambda^2 - \lambda + 1 = 0$, but this has no real roots, so this case has no points.

(c) Lagrange critical points of the boundary piece #2,

$$\{z^2 \geq x^2 + y^2,\ z^2 = 4\}$$

Case A The equations are

$$\nabla h(x,y,z) = \langle 0,0,0\rangle, \quad g(x,y,z) \geq 0,\ h(x,y,z) = 4$$

As $\nabla h(x,y,z) = \langle 0,0,2z\rangle$, it is $\langle 0,0,0\rangle$ exactly when $z = 0$. This does not satisfy $z^2 = 4$, so there are no points in this case.

Case B The equations are

$$\nabla f(x,y,z) = \lambda \nabla h(x,y,z), \quad g(x,y,z) \geq 0,\ h(x,y,z) = 4$$

The vector equations says that $\langle z-y, z-x, x+y\rangle = \lambda\langle 0,0,2z\rangle = \langle 0,0,2\lambda z\rangle$. Thus $z - y = 0$ and $z - x = 0$, or $x = y = z$. As $z^2 = 4$, $x^2 = y^2 = 4$, so this violates $z^2 \geq x^2 + y^2$, so there are no points from this case.

(d) Lagrange critical points of the boundary piece #3,

$$\{z^2 = x^2 + y^2,\ z^2 = 4\}$$

Case A1 The equations are

$$\nabla g(x,y,z) = \langle 0,0,0\rangle, \quad g(x,y,z) = 0,\ h(x,y,z) = 4$$

As $\nabla g(x,y,z) = \langle -2x, -2y, 2z\rangle$, it is $\langle 0,0,0\rangle$ if $x = y = z = 0$. This does not satisfty the domain equations, so there are no points from this case.

Case A2 The equations are

$$\nabla h(x,y,z) = \langle 0,0,0\rangle, \quad g(x,y,z) = 0,\ h(x,y,z) = 4$$

As $\nabla h(x,y,z) = \langle 0,0,2z\rangle$, it is $\langle 0,0,0\rangle$ exactly when $z = 0$. This does not satisfy $z^2 = 4$, so there are no points in this case.

Case B The equations are

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), \quad g(x,y,z)=0,\ h(x,y,z)=4$$

We get the system of equations

$$\boxed{Eq1} \cdots z - y = -2\lambda x$$

$$\boxed{Eq2} \cdots z - x = -2\lambda y$$

$$\boxed{Eq3} \cdots x + y = 2\lambda z + 2\mu z$$

$$\boxed{Eq4} \cdots z^2 = x^2 + y^2$$

$$\boxed{Eq5} \cdots z^2 = 4$$

We compute $\boxed{Eq1} - \boxed{Eq2}$ and get

$$x - y = -2\lambda(x-y)$$

So we want to divide by $(x-y)$ and get $\boxed{\lambda = -\frac{1}{2}}$, which may not be possible if $\boxed{x = y}$.

- The case of $\boxed{\lambda = -\frac{1}{2}}$. Then $\boxed{Eq1}$ says $z - y = x$, or $z = x + y$. As $z^2 = x^2 + y^2$, this means $(x+y)^2 = x^2+y^2$, or $2xy = 0$, so either $x = 0$ or $y = 0$. As from $\boxed{Eq5}$, either $z = 2$ or $z = -2$, so we get points $\boxed{(0,2,2)}$, $\boxed{(0,-2,-2)}$, $\boxed{(2,0,2)}$, $\boxed{(-2,0,-2)}$.
- The case of $\boxed{x = y}$. Plugging into $\boxed{Eq4}$ you get $z^2 = 2x^2$. As $z^2 = 4$, $x^2 = 2$, so either $x = \sqrt{2}$ or $x = -\sqrt{2}$. As $z = 2$ or $z = -2$, we get the points $\boxed{(\sqrt{2},\sqrt{2},2)}$, $\boxed{(\sqrt{2},\sqrt{2},-2)}$, $\boxed{(-\sqrt{2},-\sqrt{2},2)}$, $\boxed{(-\sqrt{2},-\sqrt{2},-2)}$,

The list of values on the candidate points:

- $f(0,0,0) = 0$
- $f(0,2,2) = 4$
- $f(0,-2,-2) = 4$
- $f(2,0,2) = 4$
- $f(-2,0,-2) = 4$
- $f(\sqrt{2},\sqrt{2},2) = 4\sqrt{2} - 2$
- $f(\sqrt{2},\sqrt{2},-2) = -4\sqrt{2} - 2$
- $f(-\sqrt{2},-\sqrt{2},2) = -4\sqrt{2} - 2$
- $f(-\sqrt{2},-\sqrt{2},-2) = 4\sqrt{2} - 2$

So the global maximum value is 4 and the global minimum value is $-4\sqrt{2} - 2$.

$\square$

Exercise 1. Determine whether the following domain is bounded or not.

(1) $\{(x,y) \mid x^2+y^2 \leq 1\}$
(2) $\{(x,y) \mid x+y=0\}$
(3) $\{(x,y) \mid x^3+y^3 \leq 1\}$
(4) $\{(x,y) \mid x^4+y^2 \leq 1\}$
(5) $\{(x,y) \mid x^2+y^4+x \leq 1,\ y \geq 0\}$
(6) $\{(x,y,z) \mid x^2+y^2+z \leq 1\}$
(7) $\{(x,y,z) \mid x^2+y^4 \leq z^2\}$
(8) $\{(x,y,z) \mid x^2+y^2+z^2 \leq 2x+2y+2z,\ z \geq 0\}$

Solution. (1) Bounded.
(2) Not bounded (x can go to $+\infty$ while y goes to $-\infty$, for example).
(3) Not bounded (x can go to $-\infty$, for example).
(4) Bounded.
(5) Bounded. This is a bit subtle, so let's explain why.
Note that $x^2+y^4+x \leq 1$ implies that $x^2+x \leq x^2+y^4+x \leq 1$, and $x^2+x \leq 1$ implies that x is bounded (it lies in between the two roots of the equation $x^2+x=1$). Now $y^4 \leq 1-(x^2+x)$, so in particular y^4 is also smaller than some fixed finite number. This implies that y is bounded too.
(6) Not bounded (z can go to $-\infty$, for example)
(7) Not bounded (z can go to $+\infty$, for example)
(8) Bounded. This is because $x^2+y^2+z^2 \leq 2x+2y+2z$ is the same as $(x^2-2x)+(y^2-2y)+(z^2-2z) \leq 0$, or $(x^2-2x+1)+(y^2-2y+1)+(z^2-2z+1) \leq 3$, or $(x-1)^2+(y-1)^2+(z-1)^2 \leq 3$, which makes the domain bounded.

$\square$

Exercise 2. Determine whether the following function is a coercive function or not.

(1) $f(x)=x^2$
(2) $f(x)=e^x$
(3) $f(x)=x^2-1$
(4) $f(x,y)=x^4+y^4$
(5) $f(x,y)=e^{x^2+y^2}$
(6) $f(x,y)=e^{x^2-y^2}$
(7) $f(x,y,z)=x^2+y^2+z^2+\sin^2(x)$

Proof.

(1) Note that $\{x^2 \leq k\}$ is empty if $k<0$ and is $\{-\sqrt{k} \leq x \leq \sqrt{k}\}$ if $k \geq 0$, so f is coercive.
(2) The domain $\{e^x \leq k\}$ is $\{x \leq \log(k)\}$, which is in general not bounded. So f is not coercive.
(3) Note that $\{x^2-1 \leq k\}$ is the same as $\{x^2 \leq k\}$. So $f(x)=x^2$ being coercive implies $f(x)=x^2-1$ is also coercive.
(4) The domain $\{x^4+y^4 \leq k\}$ is empty if $k<0$ and would imply that $x^4, y^4 \leq x^4+y^4 \leq k$, which means, if $k \geq 0$, $-\sqrt[4]{k} \leq x, y \leq \sqrt[4]{k}$, which is bounded. So f is coercive.
(5) The domain $\{e^{x^2+y^2} \leq k\}$ is the same as $\{x^2+y^2 \leq \log(k)\}$, which is bounded as x^2+y^2 is coercive. Thus f is coercive.

(6) The domain $\{e^{x^2-y^2} \le k\}$ is the same as $\{x^2 - y^2 \le \log(k)\}$, which is in general not bounded. So f is not coercive.
(7) We know $x^2 + y^2 + z^2$ is coercive. As $f(x, y, z) \ge x^2 + y^2 + z^2$, f is coercive as well.

□

Exercise 3. Determine whether f has a global maximum and/or minimum on the domain, and if they exist, find the values.

(1) $f(x, y) = x^2 + y^2$, on the domain $\{(x, y) \mid xy \ge 1\}$
(2) $f(x, y) = x^4 + y^4$ on the domain $\{(x, y) \mid x^2 - y^2 \ge 1\}$
(3) $f(x, y, z) = x^2 + y^2 + z^2$ on the domain $\{(x, y, z) \mid x - y = 1,\ y^2 - z^2 = 1\}$
(4) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ on the domain $\{(x, y, z) \mid x + y + z = 1,\ x - y + 2z = 2\}$
(5) $f(x, y, z) = x^2 + y^2 + z^2$ on the domain $\{(x, y, z) \mid 2x + y + 2z = 9,\ 5x + 5y + 7z = 29\}$
(6) $f(x, y, z) = x^2 + y^2 + z^2$ on the domain $\{(x, y, z) \mid z^2 = x^2 + y^2,\ x + y - z + 1 = 0\}$
(7) $f(x, y, z) = 2x^2 + 2y^2 + z^2 + (x-y)^2$ on the domain $\{(x, y, z) \mid xz + yz \ge 4,\ x^2 - y^2 \ge 0\}$

Solution.

(1) First we see that the domain is not bounded. On the other hand $f(x, y)$ is coercive. So we know automatically that the global maximum does not exist.

To obtain the global minimum, we do the usual process: look for the following types of points (the domain has one inequality):

- Critical points of the original domain.
- Lagrange critical points of the boundary piece #1.

$$\{(x, y) \mid xy = 1\}$$

Let $g(x, y) = xy$ so that the original domain equation is $g(x, y) \ge 1$.

- Critical points of the original domain.
  This is when $\nabla f(x, y) = \langle 0, 0\rangle$. As $\nabla f(x, y) = \langle 2x, 2y\rangle$, this happens only when $x = y = 0$. This does not satisfy $xy \ge 1$, so we don't get any point from this case.
- Lagrange critical points of the boundary piece #1.

$$\{(x, y) \mid xy = 1\}$$

Case A $\nabla g(x, y) = \langle 0, 0\rangle$
As $\nabla g(x, y) = \langle y, x\rangle$, this is equal to $\langle 0, 0\rangle$ only if $x = y = 0$, which does not satisfy $xy = 1$. Thus we get no points from this case.

Case B $\nabla f(x, y) = \lambda \nabla g(x, y)$
This means $\langle 2x, 2y\rangle = \lambda\langle y, x\rangle = \langle \lambda y, \lambda x\rangle$. Thus we get the system of equations

$$\boxed{Eq1} \quad 2x = \lambda y$$

$$\boxed{Eq2} \quad 2y = \lambda x$$

$$\boxed{Eq3} \quad xy = 1$$

Multiplying $\boxed{Eq1}$, $\boxed{Eq2}$, we get

$$4xy = \lambda^2 xy$$

Since $xy = 1$, we see $4 = \lambda^2$. Thus, either $\boxed{\lambda = 2}$ or $\boxed{\lambda = -2}$.

* The case of $\boxed{\lambda = 2}$.
  Then $\boxed{Eq1}$ says $2x = 2y$, or $x = y$. From $\boxed{Eq3}$, $xy = 1$ becomes $x^2 = 1$, so either $x = y = 1$ or $x = y = -1$. Thus we get two points, $\boxed{(1,1)}$ and $\boxed{(-1,-1)}$.
* The case of $\boxed{\lambda = -2}$.
  Then $\boxed{Eq1}$ says $2x = -2y$, or $x = -y$. From $\boxed{Eq3}$, $xy = 1$ becomes $-x^2 = 1$, which is impossible. Thus we get no points from this case.

The list of values on the candidate points:
- $f(1,1) = 2$
- $f(-1,-1) = 2$

Thus the global minimum value is 2 and the global maximum does not exist.

(2) We see that the domain is not bounded, but $f(x, y)$ is coercive. Thus we know automatically that the global maximum does not exist.

We look for the following types of points:
- Critical points of the original domain.
- Lagrange critical points of the boundary piece #1.

$$\{(x, y) \mid x^2 - y^2 = 1\}$$

Let $g(x, y) = x^2 - y^2$ so that the original domain equation is $g(x, y) \geq 1$.
- Crticial points of the original domain.
  This is when $\nabla f(x, y) = \langle 0, 0\rangle$. As $\nabla f(x, y) = \langle 4x^3, 4y^3\rangle$, this happens only when $x = y = 0$. This does not satisfy $x^2 - y^2 \geq 1$. Thus we get no points from this case.
- Lagrange critical points of the boundary piece #1.

$$\{(x, y) \mid x^2 - y^2 = 1\}$$

Case A $\nabla g(x, y) = \langle 0, 0\rangle$
As $\nabla f(x, y) = \langle 2x, -2y\rangle$, this is equal to $\langle 0, 0\rangle$ precisely if $x = y = 0$, which does not satisfy $x^2 - y^2 = 1$.

Case B $\nabla f(x, y) = \lambda \nabla g(x, y)$
This means $\langle 4x^3, 4y^3\rangle = \lambda\langle 2x, -2y\rangle$. We get a system of equations

$$\boxed{Eq1} \quad 4x^3 = 2\lambda x$$

$$\boxed{Eq2} \quad 4y^3 = -2\lambda y$$

$$\boxed{Eq3} \quad x^2 - y^2 = 1$$

We want to divide $\boxed{Eq1}$ by $2x$ and obtain $\boxed{\lambda = 2x^2}$. This may not be possible if $\boxed{x = 0}$.

* The case of $\boxed{\lambda = 2x^2}$.
  Applying the same logic to $\boxed{Eq2}$, we have either $\boxed{\lambda = -2y^2}$ or $\boxed{y = 0}$.
  · The case of $\boxed{\lambda = -2y^2}$.
  Then $2x^2 = -2y^2$. As $x^2 \geq 0$ and $-y^2 \leq 0$, this implies that $x = y = 0$. This is impossible as we need $x^2 - y^2 = 1$, so we get no points from this case.

· The case of $\boxed{y=0}$.

By $\boxed{Eq3}$, we get $x^2 = 1$, so either $x = 1$ or $x = -1$. Thus we get two points, $\boxed{(1,0)}$, $\boxed{(-1,0)}$.

∗ The case of $\boxed{x=0}$.

By $\boxed{Eq3}$, we get $-y^2 = 1$, which is impossible.

The list of values on the candidate points:
- $f(1,0) = 1$
- $f(-1,0) = 1$

Thus the global minimum value is 1, and the global maximum does not exist.

(3) We see that the domain is not bounded (z can be any number, for example), but $f(x,y,z)$ is coercive. Thus we know automatically that the global maximum does not exist.

We need to look for the Lagrange critical points of the original domain. Let $g(x,y,z) = x - y$ and $h(x,y,z) = y^2 - z^2$ so that the original domain equations are $g(x,y,z) = 1$, $h(x,y,z) = 1$.

- Lagrange critical points of the original domain.

Case A1 $\nabla g(x,y,z) = \langle 0,0,0\rangle$

As $\nabla g(x,y,z) = \langle 1,-1,0\rangle$, this is never zero.

Case A2 $\nabla h(x,y,z) = \langle 0,0,0\rangle$

As $\nabla h(x,y,z) = \langle 0,2y,-2z\rangle$, this is zero exactly when $y = z = 0$, which is impossible as $y^2 - z^2 = 1$.

Case B $\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$

As $\nabla f(x,y,z) = \langle 2x,2y,2z\rangle$, we get a system of equations

$$\boxed{Eq1} \quad 2x = \lambda$$

$$\boxed{Eq2} \quad 2y = -\lambda + 2\mu y$$

$$\boxed{Eq3} \quad 2z = -2\mu z$$

$$\boxed{Eq4} \quad x - y = 1$$

$$\boxed{Eq5} \quad y^2 - z^2 = 1$$

From $\boxed{Eq3}$, we want to divide by $-2z$ and get $\boxed{\mu=-1}$, or it is impossible if $\boxed{z=0}$.

∗ The case of $\boxed{\mu=-1}$.

From $\boxed{Eq2}$, we get $2y = -\lambda - 2y$, so $4y = -\lambda$. Thus $x = -2y$. From $x - y = 1$, we get $-3y = 1$, so $y = -\frac{1}{3}$. However from $\boxed{Eq5}$ we have $\frac{1}{9} - z^2 = 1$, so $z^2 = -\frac{8}{9}$, which is impossible.

∗ The case of $\boxed{z=0}$.

From $\boxed{Eq5}$, $y^2 = 1$, so either $y = 1$ or $y = -1$. From $x - y = 1$, we get two points, $\boxed{(0,-1,0)}$ and $\boxed{(2,1,0)}$.

The list of values on the candidate points:
- $f(0,-1,0) = 1$
- $f(2,1,0) = 5$

Thus the global minimum value is 1 and the global maximum does not exist.

(4) We see that the domain is not bounded (z can be any number, for example), but $f(x, y, z)$ is coercive (it is $\geq x^2 + y^2 + z^2$). Thus we know automatically that the global maximum does not exist.

We need to look for the Lagrange critical points of the original domain. Let $g(x, y, z) = x + y + z$ and $h(x, y, z) = x - y + 2z$ so that the domain equations are $g(x, y, z) = 1$ and $h(x, y, z) = 2$.

- Lagrange critical points of the original domain.

Case A1 $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$
As $\nabla g(x, y, z) = \langle 1, 1, 1\rangle$, it is never zero.

Case A2 $\nabla h(x, y, z) = \langle 0, 0, 0\rangle$
As $\nabla h(x, y, z) = \langle 1, -1, 2\rangle$, it is never zero.

Case B $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$
As $\nabla f(x, y, z) = \langle 2x, 4y, 6z\rangle$, we get a system of equations

$$\boxed{Eq1} \quad 2x = \lambda + \mu$$

$$\boxed{Eq2} \quad 4y = \lambda - \mu$$

$$\boxed{Eq3} \quad 6z = \lambda + 2\mu$$

$$\boxed{Eq4} \quad x + y + z = 1$$

$$\boxed{Eq5} \quad x - y + 2z = 2$$

Taking $3 \times \boxed{Eq1} - \boxed{Eq2} - 2 \times \boxed{Eq3}$, we get

$$6x - 4y - 12z = 3(\lambda + \mu) - (\lambda - \mu) - 2(\lambda + 2\mu) = 0$$

Thus $4y = 6x - 12z$, so $y = \frac{3}{2}x - 3z$. Then $\boxed{Eq4}$, $\boxed{Eq5}$ become

$$\frac{5}{2}x - 2z = 1$$

$$-\frac{1}{2}x + 5z = 2 \quad \Rightarrow \quad -\frac{5}{2}x + 25z = 10$$

Thus $23z = 11$, so $z = \frac{11}{23}$. Also as $-\frac{1}{2}x + 5z = 2$, $x = 10z - 4 = \frac{110}{23} - 4 = \frac{18}{23}$.

Thus $y = \frac{3}{2}x - 3z = \frac{27}{23} - \frac{33}{23} = -\frac{6}{23}$. Thus we get the point $\boxed{(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23})}$.

The list of values on the candidate points:

- $f(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}) = \frac{18^2 + 2\cdot 6^2 + 3\cdot 11^2}{23^2} = \frac{33}{23}$

Thus the global minimum value is $\frac{33}{23}$ and the global maximum does not exist.

(5) We see that the domain is not bounded (z can be any number, for example), but $f(x, y, z)$ is coercive. Thus we know automatically that the global maximum does not exist.

We look for the Lagrange critical points of the original domain. Let $g(x, y, z) = 2x + y + 2z$ and $h(x, y, z) = 5x + 5y + 7z$, so the domain equations are $g(x, y, z) = 9$ and $h(x, y, z) = 29$.

- Lagrange critical points of the original domain.

Case A1 $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$
As $\nabla g(x, y, z) = \langle 2, 1, 2\rangle$, it is never zero.

Case A2 $\nabla h(x,y,z) = \langle 0,0,0\rangle$

As $\nabla h(x,y,z) = \langle 5,5,7\rangle$, it is never zero.

Case B $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

As $\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$, we get a system of equations

$$\boxed{Eq1} \quad 2x = 2\lambda + 5\mu$$

$$\boxed{Eq2} \quad 2y = \lambda + 5\mu$$

$$\boxed{Eq3} \quad 2z = 2\lambda + 7\mu$$

$$\boxed{Eq4} \quad 2x + y + 2z = 9$$

$$\boxed{Eq5} \quad 5x + 5y + 7z = 29$$

Taking $\boxed{Eq1} - \boxed{Eq2}$, we get $2x - 2y = \lambda$. Taking $\boxed{Eq3} - \boxed{Eq1}$, we get $2z - 2x = 2\mu$, or $z - x = \mu$. So $\boxed{Eq2}$ says $2y = (2x-2y)+5(z-x) = -3x-2y+5z$, or $3x + 4y - 5z = 0$. Taking $\boxed{Eq5} - \boxed{Eq4}$, we get $3x + 4y + 5z = 20$. So $10z = 20$, or $z = 2$. Thus we get

$$\boxed{Eq4} \quad 2x + y = 5$$

$$\boxed{Eq5} \quad 5x + 5y = 15 \quad \Rightarrow \quad x + y = 3$$

So $x = 2$ and $y = 1$, and we get the piont $\boxed{(2,1,2)}$.

The list of values on the candidate points:

- $f(2,1,2) = 9$

Thus the global minimum value is 9 and the global maximum does not exist.

(6) Let's first see whether the domain is bounded or not, which is unclear at first sight. Plugging $z = x + y + 1$ into $z^2 = x^2 + y^2$, we get

$$x^2 + y^2 + 1 + 2xy + 2x + 2y = x^2 + y^2$$

or

$$2xy + 2x + 2y + 1 = 0$$

This means $2xy + 2x + 2y + 2 = 1$, or $2(x+1)(y+1) = 1$. So definitely x can be any number $\neq = -1$, which makes the domain unbounded. On the other hand, $f(x,y,z)$ is coercive. Thus we know automatically that the global maximum does not exist.

We look for the Lagrange critical points of the original domain. Let $g(x,y,z) = x^2 + y^2 - z^2$ and $h(x,y,z) = x + y - z + 1$ so that the domain equations are $g(x,y,z) = 0$ and $h(x,y,z) = 0$.

- Lagrange critical points of the original domain.

Case A1 $\nabla g(x,y,z) = \langle 0,0,0\rangle$

As $\nabla g(x,y,z) = \langle 2x, 2y, -2z\rangle$, it is equal to zero precisely if $x = y = z = 0$. However this does not satisfy $x + y - z + 1 = 0$, so there is no point from this case.

Case A2 $\nabla h(x,y,z) = \langle 0,0,0\rangle$

As $\nabla h(x,y,z) = \langle 1,1,-1\rangle$, it is never zero.

Case B $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$
As $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$, we get a system of equations

$$\boxed{Eq1} \quad 2x = 2\lambda x + \mu$$

$$\boxed{Eq2} \quad 2y = 2\lambda y + \mu$$

$$\boxed{Eq3} \quad 2z = -2\lambda z - \mu$$

$$\boxed{Eq4} \quad x^2 + y^2 - z^2 = 0$$

$$\boxed{Eq5} \quad x + y - z + 1 = 0$$

Taking $\boxed{Eq1} - \boxed{Eq2}$, we get $2(x-y) = 2\lambda(x-y)$. We want to divide it by $2(x-y)$ and get $\boxed{\lambda = 1}$, which may not be possible if $\boxed{x = y}$.

* The case of $\boxed{\lambda = 1}$.
  Then $\boxed{Eq1}$ implies $2x = 2x + \mu$, so $\mu = 0$. Thus $\boxed{Eq3}$ means $2z = -2z$, so $z = 0$. Thus $\boxed{Eq4}$ beeomse $x^2 + y^2 = 0$, so $x = y = 0$. This does not satisfy $\boxed{Eq5}$, so we get no points in this case.
* The case of $\boxed{x = y}$.
  Then $\boxed{Eq4}$ becomes $2x^2 = z^2$, which means either $\boxed{z = \sqrt{2}x}$ or $\boxed{z = -\sqrt{2}x}$.
  · The case of $\boxed{z = \sqrt{2}x}$.
  Then $\boxed{Eq5}$ becomes $(2-\sqrt{2})x = -1$, or $x = -\frac{1}{2-\sqrt{2}} = -\frac{2+\sqrt{2}}{2} = -\frac{\sqrt{2}+1}{\sqrt{2}}$. Thus we get a point $\boxed{(-\dfrac{\sqrt{2}+1}{\sqrt{2}}, -\dfrac{\sqrt{2}+1}{\sqrt{2}}, -(\sqrt{2}+1))}$.
  · The case of $\boxed{z = -\sqrt{2}x}$.
  Then $\boxed{Eq5}$ becomes $(2+\sqrt{2})x = -1$, or $x = -\frac{1}{2+\sqrt{2}} = -\frac{2-\sqrt{2}}{2} = -\frac{\sqrt{2}-1}{\sqrt{2}}$. Thus we get a point $\boxed{(-\dfrac{\sqrt{2}-1}{\sqrt{2}}, -\dfrac{\sqrt{2}-1}{\sqrt{2}}, -(\sqrt{2}-1))}$.

The list of values on the candidate points:

- $f(-\frac{\sqrt{2}+1}{\sqrt{2}}, -\frac{\sqrt{2}+1}{\sqrt{2}}, -(\sqrt{2}+1)) = \frac{(\sqrt{2}+1)^2}{2} + \frac{(\sqrt{2}+1)^2}{2} + (\sqrt{2}+1)^2 = 2(\sqrt{2}+1)^2 = 6+4\sqrt{2}$
- $f(-\frac{\sqrt{2}-1}{\sqrt{2}}, -\frac{\sqrt{2}-1}{\sqrt{2}}, -(\sqrt{2}-1)) = \frac{(\sqrt{2}-1)^2}{2} + \frac{(\sqrt{2}-1)^2}{2} + (\sqrt{2}-1)^2 = 2(\sqrt{2}-1)^2 = 6-4\sqrt{2}$

Thus the global minimum value is $6-4\sqrt{2}$ and the global maximum does not exist.

(7) We see that the domain is not bounded ($(1,0,z)$ is in the domain for $z \geq 4$, for example), but $f(x,y,z)$ is coercive (it is $\geq x^2+y^2+z^2$). Thus we know automatically that the global maximum does not exist.

We look for the following types of points:

- Critical points of the original domain.
- Lagrange critical points of the boundary piece #1.

$$\{(x,y,z) \mid xz + yz = 4,\ x^2 - y^2 \geq 0\}$$

- Lagrange critical points of the boundary piece #2.
$$\{(x, y, z) \mid xz + yz \geq 4,\ x^2 - y^2 = 0\}$$
- Lagrange critical points of the boundary piece #3.
$$\{(x, y, z) \mid xz + yz = 4,\ x^2 - y^2 = 0\}$$

Let $g(x, y, z) = xz + yz$ and $h(x, y, z) = x^2 - y^2$, so that the domain equations are $g(x, y, z) \geq 4$ and $h(x, y, z) \geq 0$.

- Critical points of the original domain.
  This means $\nabla f(x, y, z) = \langle 0, 0, 0\rangle$. As $\nabla f(x, y, z) = \langle 4x + 2(x - y), 4y + 2(y - x), 2z\rangle = \langle 6x-2y, -2x+6y, 2z\rangle$, this being $\langle 0, 0, 0\rangle$ means $6x-2y = 0$, $-2x+6y = 0$, and $2z = 0$. So, $z = 0$, $y = 3x$ and $x = 3y$, so $x = 9x$, which means $x = 0$, so $y = 0$. This does not satisfy $g(x, y, z) \geq 4$, so we get no points from this case.
- Lagrange critical points of the boundary piece #1.
$$\{(x, y, z) \mid xz + yz = 4,\ x^2 - y^2 \geq 0\}$$

Case A $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$
  As $\nabla g(x, y, z) = \langle z, z, x + y\rangle$, this being zero means $z = 0$, which does not satisfy $xz + yz = 4$, so we get no points from this case.

Case B $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
  We get a system of equations
$$\boxed{Eq1} \quad 6x - 2y = \lambda z$$
$$\boxed{Eq2} \quad -2x + 6y = \lambda z$$
$$\boxed{Eq3} \quad 2z = \lambda(x + y)$$
$$\boxed{Eq4} \quad xz + yz = 4$$
$$\boxed{Eq5} \quad x^2 - y^2 \geq 0$$
  By comparing $\boxed{Eq1}$ and $\boxed{Eq2}$, we get $6x - 2y = -2x + 6y$, or $8x = 8y$, or $x = y$. This gives
$$\boxed{Eq1} \quad 4x = \lambda z$$
$$\boxed{Eq3} \quad 2z = 2\lambda x \quad \Rightarrow \quad z = \lambda x$$
$$\boxed{Eq4} \quad 2xz = 4 \quad \Rightarrow \quad xz = 2$$
  Taking $\boxed{Eq1} \times \boxed{Eq3}$, we get $4xz = \lambda^2 xz$. As $xz = 2$, $\lambda^2 = 4$, so either $\lambda = 2$ or $\lambda = -2$. On the other hand, $\boxed{Eq3}$ plugged into $\boxed{Eq4}$ gives $\lambda x^2 = 2$, so this implies that λ cannot be negative, so $\lambda = 2$, and $x^2 = 1$. Thus either $x = 1$ or $x = -1$. Thus we get two points, $\boxed{(1, 1, 2)}$, $\boxed{(-1, -1, -2)}$.
- Lagrange critical points of the boundary piece #2.
$$\{(x, y, z) \mid xz + yz \geq 4,\ x^2 - y^2 = 0\}$$

Case A $\nabla h(x, y, z) = \langle 0, 0, 0\rangle$
  As $\nabla h(x, y, z) = \langle 2x, -2y, 0\rangle$, this being zero means $x = y = 0$, which does not satisfy $xz + yz \geq 4$, so we get no points from this case.

Case B $\nabla f(x, y, z) = \lambda \nabla h(x, y, z)$

We get a system of equations

$$\boxed{Eq1} \quad 6x - 2y = 2\lambda x$$

$$\boxed{Eq2} \quad -2x + 6y = -2\lambda y$$

$$\boxed{Eq3} \quad 2z = 0$$

$$\boxed{Eq4} \quad xz + yz \geq 4$$

$$\boxed{Eq5} \quad x^2 - y^2 = 0$$

As $\boxed{Eq3}$ implies $z = 0$, this violates $\boxed{Eq4}$, giving no points in this case.

- Lagrange critical points of the boundary piece #3.

$$\{(x, y, z) \mid xz + yz = 4,\ x^2 - y^2 = 0\}$$

Case A1 $\nabla g(x, y, z) = \langle 0, 0, 0 \rangle$

As $\nabla g(x, y, z) = \langle z, z, x + y \rangle$, this being zero means $z = 0$, which does not satisfy $xz + yz = 4$, so we get no points from this case.

Case A2 $\nabla h(x, y, z) = \langle 0, 0, 0 \rangle$

As $\nabla h(x, y, z) = \langle 2x, -2y, 0 \rangle$, this being zero means $x = y = 0$, which does not satisfy $xz + yz = 4$, so we get no points from this case.

Case B $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$

We get a system of equations

$$\boxed{Eq1} \quad 6x - 2y = \lambda z + 2\mu x$$

$$\boxed{Eq2} \quad -2x + 6y = \lambda z - 2\mu y$$

$$\boxed{Eq3} \quad 2z = \lambda(x + y)$$

$$\boxed{Eq4} \quad xz + yz = 4$$

$$\boxed{Eq5} \quad x^2 - y^2 = 0$$

From $\boxed{Eq5}$, $x^2 = y^2$, so either $x = y$ or $x = -y$. Since $\boxed{Eq4}$ means $(x + y)z = 4$, this means $x = -y$ is not possible. Thus $x = y$. Thus, the system becomes

$$\boxed{Eq1} \quad 4x = \lambda z + 2\mu x$$

$$\boxed{Eq2} \quad 4x = \lambda z - 2\mu x$$

$$\boxed{Eq3} \quad 2z = 2\lambda x$$

$$\boxed{Eq4} \quad 2xz = 4$$

By comparing $\boxed{Eq1}$ and $\boxed{Eq2}$, we get $2\mu x = 0$. This means either $\mu = 0$ or $x = 0$. However, $\boxed{Eq4}$ means that $xz = 2$, which means $x = 0$ is impossible. Thus, we have $\mu = 0$. Thus the system becomes

$$\boxed{Eq1} \quad 4x = \lambda z$$

$$\boxed{Eq3} \quad 2z = 2\lambda x$$

$$\boxed{Eq4} \quad 2xz = 4$$

This is exactly the system appeared in the Case B of boundary piece #1, so there are no new points.

The list of values on the candidate points:

- $f(1, 1, 2) = 8$
- $f(-1, -1, -2) = 8$

So the global minimum value is 8, and the global maximum does not exist.

$\square$

Exercise 4. Find all the points on the plane $x + y + z = 1$ that are closest to the point $(2, 0, -3)$ and compute the distance.

Solution. We know that we just need to look for the smallest value of $f(x, y, z) = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$ on the Lagrange critical points. Since $f(x, y, z)$ is coercive, we just do the usual Lagrange critical points of the original domain. Let $g(x, y, z) = x + y + z$ so that the domain equation is $g(x, y, z) = 1$.

Case A $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$

As $\nabla g(x, y, z) = \langle 1, 1, 1\rangle$, it is never zero.

Case B $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

We get a system of equations

$$\boxed{Eq1} \quad \frac{x-2}{\sqrt{(x-2)^2 + y^2 + (z+3)^2}} = \lambda$$

$$\boxed{Eq2} \quad \frac{y}{\sqrt{(x-2)^2 + y^2 + (z+3)^2}} = \lambda$$

$$\boxed{Eq3} \quad \frac{z+3}{\sqrt{(x-2)^2 + y^2 + (z+3)^2}} = \lambda$$

$$\boxed{Eq4} \quad x + y + z = 1$$

By comparing the numerators of the left sides of $\boxed{Eq1}$, $\boxed{Eq2}$, $\boxed{Eq3}$, we get $x - 2 = y = z+3$. This implies that $y = x-2$ and $z = x-5$. Then $\boxed{Eq4}$ becomes $x+(x-2)+(x-5) = 1$, so $3x - 7 = 1$, so $x = \frac{8}{3}$, which implies that $y = \frac{2}{3}$ and $z = -\frac{7}{3}$. Thus we get a point $\boxed{(\frac{8}{3}, \frac{2}{3}, -\frac{7}{3})}$.

We see that we only get one point, $(\frac{8}{3}, \frac{2}{3}, -\frac{7}{3})$, so this ought to be the closest point, and the distance is $f(\frac{8}{3}, \frac{2}{3}, -\frac{7}{3}) = \sqrt{3 \cdot \left(\frac{2}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$. $\square$

Exercise 5. Find all the points on the plane $x - 2y + 3z = 6$ that are closest to the point $(0, 1, 1)$ and compute the distance.

Solution. We know that we just need to look for the smallest value of $f(x, y, z) = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$ on the Lagrange critical points. Since $f(x, y, z)$ is coercive, we just do the usual Lagrange critical points of the original domain. Let $g(x, y, z) = x - 2y + 3z$ so that the domain equation is $g(x, y, z) = 6$.

Case A $\nabla g(x,y,z) = \langle 0,0,0\rangle$

As $\nabla g(x,y,z) = \langle 1,-2,3\rangle$, it is never zero.

Case B $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

We get a system of equations

$$\boxed{Eq1} \quad \frac{x}{\sqrt{x^2+(y-1)^2+(z-1)^2}} = \lambda$$

$$\boxed{Eq2} \quad \frac{y-1}{\sqrt{x^2+(y-1)^2+(z-1)^2}} = -2\lambda$$

$$\boxed{Eq3} \quad \frac{z-1}{\sqrt{x^2+(y-1)^2+(z-1)^2}} = 3\lambda$$

$$\boxed{Eq4} \quad x-2y+3z=6$$

By comparing $\boxed{Eq1}$, $-\frac{1}{2}\times\boxed{Eq2}$ and $\frac{1}{3}\boxed{Eq3}$, we have

$$\frac{x}{\sqrt{x^2+(y-1)^2+(z-1)^2}} = -\frac{y-1}{2\sqrt{x^2+(y-1)^2+(z-1)^2}} = \frac{z-1}{3\sqrt{x^2+(y-1)^2+(z-1)^2}}$$

so

$$x = -\frac{y-1}{2} = \frac{z-1}{3}$$

Thus $2x = 1-y$, or $y = 1-2x$. Also, $3x = z-1$, so $z = 3x+1$. Thus $\boxed{Eq4}$ becomes $x - 2(1-2x) + 3(3x+1) = 6$, or $x + 4x - 2 + 9x + 3 = 6$, or $14x + 1 = 6$, or $x = \frac{5}{14}$. Thus $y = \frac{2}{7}$, and $z = \frac{29}{14}$. Thus, we get a point $\boxed{(\frac{5}{14}, \frac{2}{7}, \frac{29}{14})}$.

We see that we only get one point, $(\frac{5}{14}, \frac{2}{7}, \frac{29}{14})$, so this must be the closest point, and the distance is

$$f(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}) = \sqrt{\left(\frac{5}{14}\right)^2 + \left(-\frac{5}{7}\right)^2 + \left(\frac{15}{14}\right)^2} = \frac{5}{14}\sqrt{1+4+9} = \frac{5}{\sqrt{14}}$$

□

Exercise 6. Find all the points on the surface $z = x^2 + y^2$ that are closest to the point $(5,5,0)$ and compute the distance.

Solution. We know that we just need to look for the smallest value of $f(x,y,z) = \sqrt{(x-5)^2+(y-5)^2+z^2}$ on the Lagrange critical points. Since $f(x,y,z)$ is coercive, we just do the usual Lagrange critical points of the original domain. Let $g(x,y,z) = x^2+y^2-z$ so that the domain equation is $g(x,y,z) = 0$.

Case A $\nabla g(x,y,z) = \langle 0,0,0\rangle$

As $\nabla g(x,y,z) = \langle 2x, 2y, -1\rangle$, this is never zero, as the z component is never zero.

Case B $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

We get a system of equations

$$\boxed{Eq1} \quad \frac{x-5}{\sqrt{(x-5)^2+(y-5)^2+z^2}} = 2\lambda x$$

$$\boxed{Eq2} \quad \frac{y-5}{\sqrt{(x-5)^2+(y-5)^2+z^2}} = 2\lambda y$$

$$\boxed{Eq3} \quad \frac{z}{\sqrt{(x-5)^2+(y-5)^2+z^2}} = -\lambda$$

$$\boxed{Eq4} \quad z = x^2 + y^2$$

By plugging $\boxed{Eq3}$ into $\boxed{Eq1}$, $\boxed{Eq2}$, we get

$$\boxed{Eq1} \quad \frac{x-5}{\sqrt{(x-5)^2+(y-5)^2+z^2}} = -\frac{2xz}{\sqrt{(x-5)^2+(y-5)^2+z^2}}$$

$$\boxed{Eq2} \quad \frac{y-5}{\sqrt{(x-5)^2+(y-5)^2+z^2}} = -\frac{2yz}{\sqrt{(x-5)^2+(y-5)^2+z^2}}$$

so

$$x-5 = -2xz, \quad y-5 = -2yz$$

From this, we see that $x=0$ is impossible (the first equation would be $-5=0$), and similarly $y=0$ is impossible. Thus, we can divide the first equation by x and the second equation by y and get

$$1-\frac{5}{x} = \frac{x-5}{x} = -2z = \frac{y-5}{y} = 1-\frac{5}{y}$$

Thus, $\frac{5}{x} = \frac{5}{y}$, so $x=y$. Thus $\boxed{Eq4}$ means $z = 2x^2$, and we also have $x-5 = -2xz = -4x^3$, which is $4x^3+x-5=0$. As $x=1$ is an obvious solution, this factors into $(x-1)(4x^2+4x+5)=0$. Since $4x^2+4x+5$ has no real solution, we see that $x=1$ is the only solution to $4x^3+x-5=0$. Thus we get one point, $\boxed{(1,1,2)}$.

As we have only gotten one point $(1,1,2)$, this must be the closest point, and the distance is $f(1,1,2) = \sqrt{4^2+4^2+2^2} = \sqrt{36} = 6$. □

Exercise 7. Find all the points on the surface $z = x^2+y^2$ that are closest to the point $(1,1,0)$ and compute the distance.

Solution. We know that we just need to look for the smallest value of $f(x,y,z) = \sqrt{(x-1)^2+(y-1)^2+z^2}$ on the Lagrange critical points. Since $f(x,y,z)$ is coercive, we just do the usual Lagrange critical points of the original domain. Let $g(x,y,z) = x^2+y^2-z$ so that the domain equation is $g(x,y,z)=0$.

Case A $\nabla g(x,y,z) = \langle 0,0,0\rangle$

As $\nabla g(x,y,z) = \langle 2x, 2y, -1\rangle$, this is never zero, as the z component is never zero.

Case B $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

We get a system of equations

$$\boxed{Eq1} \quad \frac{x-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = 2\lambda x$$

$$\boxed{Eq2} \quad \frac{y-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = 2\lambda y$$

$$\boxed{Eq3} \quad \frac{z}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = -\lambda$$

$$\boxed{Eq4} \quad z = x^2+y^2$$

By plugging $\boxed{Eq3}$ into $\boxed{Eq1}$, $\boxed{Eq2}$, we get

$$\boxed{Eq1} \quad \frac{x-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = -\frac{2xz}{\sqrt{(x-1)^2+(y-1)^2+z^2}}$$

$$\boxed{Eq2} \quad \frac{y-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = -\frac{2yz}{\sqrt{(x-1)^2+(y-1)^2+z^2}}$$

so

$$x-1 = -2xz, \quad y-1 = -2yz$$

From this, we see that $x = 0$ is impossible (the first equation would be $-1 = 0$), and similarly $y = 0$ is impossible. Thus, we can divide the first equation by x and the second equation by y and get

$$1 - \frac{1}{x} = \frac{x-1}{x} = -2z = \frac{y-1}{y} = 1 - \frac{1}{y}$$

Thus, $\frac{1}{x} = \frac{1}{y}$, so $x = y$. Thus $\boxed{Eq4}$ means $z = 2x^2$, and we also have $x - 1 = -2xz = -4x^3$, or $4x^3 + x - 1 = 0$. Note that $x = \frac{1}{2}$ is a solution, so it factors into $4x^3 + x - 1 = (2x-1)(2x^2+x+1)$. Since $2x^2 + x + 1$ has no real solution, the only solution of $4x^3 + x - 1 = 0$ is $x = \frac{1}{2}$. Thus we get a point $\boxed{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}$.

As we got only one point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, this must be the closest point, and the distance is $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$. □

Exercise 8. Find all the points on the surface $xy^2z^3 = 2$ that are closest to the point $(0, 0, 0)$ and compute the distance.

Solution. We know that we just need to look for the smallest value of $f(x, y, z) = \sqrt{x^2+y^2+z^2}$ on the Lagrange critical points. Since $f(x, y, z)$ is coercive, we just do the usual Lagrange critical points of the original domain. Let $g(x, y, z) = xy^2z^3$ so that the domain equation is $g(x, y, z) = 2$.

Case A $\nabla g(x, y, z) = \langle 0, 0, 0 \rangle$

As $\nabla g(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$, this being zero means $y^2z^3 = 0$, which means $xy^2z^3 = 0$, which cannot be possible under $xy^2z^3 = 2$.

Case B $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

We get a system of equations

$$\boxed{Eq1} \quad \frac{x}{\sqrt{x^2+y^2+z^2}} = \lambda y^2 z^3$$

$$\boxed{Eq2} \quad \frac{y}{\sqrt{x^2+y^2+z^2}} = 2\lambda xyz^3$$

$$\boxed{Eq3} \quad \frac{z}{\sqrt{x^2+y^2+z^2}} = 3\lambda xy^2z^2$$

$$\boxed{Eq4} \quad xy^2z^3 = 2$$

By comparing $x \times \boxed{Eq1}$, $\frac{y}{2} \times \boxed{Eq2}$, $\frac{z}{3} \times \boxed{Eq3}$, we get

$$\frac{x^2}{\sqrt{x^2+y^2+z^2}} = \frac{y^2}{2\sqrt{x^2+y^2+z^2}} = \frac{z^2}{3\sqrt{x^2+y^2+z^2}}$$

or

$$x^2 = \frac{y^2}{2} = \frac{z^2}{3}$$

This means that $y^2 = 2x^2$, or $y = \pm\sqrt{2}x$, and $z^2 = 3x^2$, or $z = \pm\sqrt{3}x$. Then $\boxed{Eq4}$ becomes $x \cdot (2x^2) \cdot (\pm\sqrt{3}x)^3 = 2$, or $\pm 3\sqrt{3}x^6 = 1$ (the sign of this is the same as the sign of $z = \pm\sqrt{3}x$). Thus $z = \sqrt{3}x$, and $3\sqrt{3}x^6 = 1$. Thus $x^6 = \frac{1}{3^{3/2}}$, so $x = \pm\frac{1}{3^{1/4}}$.

Thus we get four points, $\boxed{(\frac{1}{3^{1/4}}, \frac{\sqrt{2}}{3^{1/4}}, \frac{\sqrt{3}}{3^{1/4}})}$, $\boxed{(\frac{1}{3^{1/4}}, -\frac{\sqrt{2}}{3^{1/4}}, \frac{\sqrt{3}}{3^{1/4}})}$, $\boxed{(-\frac{1}{3^{1/4}}, \frac{\sqrt{2}}{3^{1/4}}, -\frac{\sqrt{3}}{3^{1/4}})}$, $\boxed{(-\frac{1}{3^{1/4}}, -\frac{\sqrt{2}}{3^{1/4}}, -\frac{\sqrt{3}}{3^{1/4}})}$.

The list of values on the candidate points:

- $f(\frac{1}{3^{1/4}}, \frac{\sqrt{2}}{3^{1/4}}, \frac{\sqrt{3}}{3^{1/4}}) = \sqrt{3^{-1/2} + 2 \cdot 3^{-1/2} + 3 \cdot 3^{-1/2}} = \sqrt{6 \cdot 3^{-1/2}} = \sqrt{2\sqrt{3}}$
- $f(\frac{1}{3^{1/4}}, -\frac{\sqrt{2}}{3^{1/4}}, \frac{\sqrt{3}}{3^{1/4}}) = \sqrt{3^{-1/2} + 2 \cdot 3^{-1/2} + 3 \cdot 3^{-1/2}} = \sqrt{6 \cdot 3^{-1/2}} = \sqrt{2\sqrt{3}}$
- $f(-\frac{1}{3^{1/4}}, \frac{\sqrt{2}}{3^{1/4}}, -\frac{\sqrt{3}}{3^{1/4}}) = \sqrt{3^{-1/2} + 2 \cdot 3^{-1/2} + 3 \cdot 3^{-1/2}} = \sqrt{6 \cdot 3^{-1/2}} = \sqrt{2\sqrt{3}}$
- $f(-\frac{1}{3^{1/4}}, -\frac{\sqrt{2}}{3^{1/4}}, -\frac{\sqrt{3}}{3^{1/4}}) = \sqrt{3^{-1/2} + 2 \cdot 3^{-1/2} + 3 \cdot 3^{-1/2}} = \sqrt{6 \cdot 3^{-1/2}} = \sqrt{2\sqrt{3}}$

Thus, there are four closest points, $(\frac{1}{3^{1/4}}, \frac{\sqrt{2}}{3^{1/4}}, \frac{\sqrt{3}}{3^{1/4}})$, $(\frac{1}{3^{1/4}}, -\frac{\sqrt{2}}{3^{1/4}}, \frac{\sqrt{3}}{3^{1/4}})$, $(-\frac{1}{3^{1/4}}, \frac{\sqrt{2}}{3^{1/4}}, -\frac{\sqrt{3}}{3^{1/4}})$, $(-\frac{1}{3^{1/4}}, -\frac{\sqrt{2}}{3^{1/4}}, -\frac{\sqrt{3}}{3^{1/4}})$, and the distance is $\sqrt{2\sqrt{3}}$. $\square$